Physics 5583. Electrodynamics II.
First Midterm Examination
Spring 2011

March 9, 2011

Instructions: This examination consists of three problems. If you get stuck on one part, assume a result and proceed onward. Do not hesitate to ask questions. GOOD LUCK!

1. (a) Write down Maxwell's equations in Heaviside-Lorentz units for empty space but in the presence of electric charges and currents (no magnetic charge or current).

   (b) Show that the two homogeneous Maxwell equations are automatically satisifed if a vector and a scalar potential are introduced, $\mathbf{A}$ and ϕ.

   (c) Suppose we choose a gauge in which the scalar potential ϕ is identically zero. What equation does $\boldsymbol{\nabla}\cdot\mathbf{A}$ satisfy? What wave equation does the vector potential $\mathbf{A}$ satisfy?

   (d) Show that if we write

   $$\mathbf{A} = \mathbf{A}' - \boldsymbol{\nabla}\psi,$$

   we recover the usual construction of $\mathbf{E}$ and $\mathbf{B}$ in terms of new vector and scalar potentials $\mathbf{A}'$, ϕ'. What is the relation between ψ and ϕ'?

2. (a) A particle with charge e moves along the z axis with constant speed v. Its coordinates are

   $$x(t) = 0, \quad y(t) = 0, \quad z(t) = vt.$$

Construct the potentials in the Lorenz gauge by solving the differential equations satisfied by the potentials, noting that the only variables are x, y, and $z - vt$. Show that the result for the scalar potential is

$$\phi(x, y, z, t) = \frac{e/4\pi}{\sqrt{(z - vt)^2 + (1 - v^2/c^2)(x^2 + y^2)}}.$$

What is $\mathbf{A}$?

(b) Show that the above result follows from the Liénard-Wiechert potential

$$\phi(\mathbf{r}, t) = \frac{e}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}(\tau)| - \mathbf{v}(\tau) \cdot (\mathbf{r} - \mathbf{r}(\tau))/c},$$

where the retarded time τ is given by the implicit equation

$$t - \tau = \frac{1}{c}|\mathbf{r} - \mathbf{r}(\tau)|.$$

[Hint: Solve the resulting quadratic equation for τ.]

(c) What is the power radiated by a particle moving with uniform velocity?

3. A nonrelativistic electron of charge e and mass m moves in a circular orbit under Coulomb forces produced by a proton. The average potential energy is related to the total energy by the virial theorem which here says

$$E = \frac{1}{2}\overline{V}.$$

Suppose, as it radiates, the electron continues to move in a circular orbit, of gradually decreasing radius, and calculate the power radiated, and thereby $-dE/dt$, as a function of E. The radiated power is given by the Lamor dipole-radiation formula,

$$P = \frac{1}{6\pi c^3}(\ddot{\mathbf{d}})^2.$$

Integrate this result, and find how long it takes for the energy to change from E_1 to E_2. In a finite time the electron reaches the center, so calculate how long it takes the electron to hit the proton if it starts from an initial energy of

$$E_{\text{Bohr}} = -13.6 \text{ eV}.$$

Use the following values:

$$\alpha = \frac{e^2}{4\pi\hbar c} = \frac{1}{137},$$

$$\hbar = 6.58 \times 10^{-16} \text{eV s},$$

$$mc^2 = 0.511 \text{ MeV}.$$

An order of magnitude estimate is sufficient. This instability was one of the reasons for the discovery of quantum mechanics.