Chapter 5

Non-Abelian Gauge Fields

The simplest example starts with two Fermions (Dirac particles) $\psi_1, \psi_2$, degenerate in mass, and hence satisfying in the absence of interactions

$$ (\gamma^\alpha_1 \partial + m)\psi_1 = 0, \quad (\gamma^\alpha_1 \partial + m)\psi_2 = 0. \quad (5.1) $$

We can define a two-component object $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ with the associated action

$$ W = \int (dx) \mathcal{L}, \quad \mathcal{L} = -\psi^\dagger \left( \gamma^\alpha_1 \partial + m \right) \psi. \quad (5.2) $$

If $U$ is a constant $2 \times 2$ matrix, $\mathcal{L}$ is invariant under the replacement $\psi \to U\psi$, provided $U$ is unitary,

$$ U^\dagger U = UU^\dagger = 1. \quad (5.3) $$

We know that the most general unitary $2 \times 2$ matrix, apart from a pure phase factor [$U(1)$ transformation], can be written in terms of the Pauli matrices $\tau$ as

$$ U = e^{i\lambda \cdot \tau} = \cos |\lambda| + i \frac{\lambda \cdot \tau}{|\lambda|} \sin |\lambda| $$

$$ = \begin{pmatrix} \cos \lambda + i \hat{\lambda}_3 \sin \lambda & (i\hat{\lambda}_1 + \hat{\lambda}_2) \sin \lambda \\ (i\hat{\lambda}_1 - \hat{\lambda}_2) \sin \lambda & \cos \lambda - i \hat{\lambda}_3 \sin \lambda \end{pmatrix}, \quad (5.4) $$

where $\lambda$ is an arbitrary vector. The generators of these transformations are the Pauli matrices $\tau$ which obey the algebra

$$ [\tau^a, \tau^b] = 2i\epsilon^{abc}\tau^c. \quad (5.5) $$

Because

$$ \det U = \cos^2 \lambda + (\hat{\lambda}_3^2 + \hat{\lambda}_1^2 + \hat{\lambda}_2^2) \sin^2 \lambda = 1, \quad (5.6) $$

this actually represents an SU(2) transformation.
Now suppose we gauge the symmetry, by letting $\lambda \rightarrow \lambda(x)$. Then $L$ is not invariant, 

$$\delta L = -\bar{\psi}U^\dagger \gamma^\mu \frac{1}{i} (\partial_\mu U) \psi \approx -\partial_\mu \delta \lambda \cdot \bar{\psi} \gamma^\mu \tau \psi,$$

(5.7)

for $\delta \lambda$ infinitesimal. We see here the conserved “isospin” current [compare (2.33)]

$$\delta W = -2 \int (dx) \partial_\mu \delta \lambda \cdot j^\mu = 0,$$

(5.8)

where

$$j^\mu = \frac{1}{2} \bar{\psi} \gamma^\mu \tau \psi,$$

(5.9)

which is conserved, by the stationary action principle,

$$\partial_\mu j^\mu = 0.$$

(5.10)

How can we cancel $\delta L$ identically? Evidently, by coupling to this current a triplet of vector fields,

$$A_\mu = (A_1^\mu, A_2^\mu, A_3^\mu),$$

(5.11)

as follows:

$$\mathcal{L}_\text{int}^f = g A_\mu \cdot \bar{\psi} \gamma^\mu \tau \psi,$$

(5.12)

where under an infinitesimal gauge transformation

$$A_\mu \rightarrow A_\mu + \partial_\mu \delta \omega,$$

(5.13)

where $\delta \omega$ is related to $\delta \lambda$:

$$\delta \lambda = g \frac{2}{\tau} \delta \omega.$$

(5.14)

But this is not the whole story! That is because the $\psi$ variation of $\mathcal{L}_\text{int}^f$ is

$$\delta_\psi \mathcal{L}_\text{int}^f = -\bar{\psi} \gamma^\mu \left[ \frac{\tau}{2} \cdot g A_\mu, \frac{i}{2} g \tau \cdot \delta \omega \right] \psi$$

$$= g \bar{\psi} \gamma^\mu \frac{\tau}{2} \cdot (A_\mu \times \delta \omega) \psi \neq 0.$$

(5.15)

This will be cancelled if we modify our $A_\mu$ variation to

$$A_\mu \rightarrow A_\mu + \partial_\mu \delta \omega - g \delta \omega \times A_\mu.$$

(5.16)

This last is in fact the transformation law for a vector under an ordinary rotation. Mathematically, we say that these gauge fields transform as the spin-1 (adjoint) representation of SU(2).

Now assemble the Fermion parts of $\mathcal{L}$:

$$\mathcal{L}_f = -\bar{\psi} \left( \gamma^\mu \left( \frac{1}{i} D + m \right) \right) \psi,$$

(5.17)
where the gauge covariant derivative is

\[ D_\mu = \partial_\mu - \frac{i}{2} g \tau \cdot A_\mu. \]  

(5.18)

Under a gauge transformation,

\[ D_\mu \rightarrow U^\dagger D_\mu U = D_\mu. \]  

(5.19)

That is,

\[ D_\mu U^\dagger = \partial_\mu - \frac{ig}{2} \tau \cdot A_\mu U^\dagger = U (\partial_\mu - \frac{ig}{2} \tau \cdot A_\mu) U^\dagger, \]

which says

\[ \tau \cdot A_\mu U^\dagger = U \tau \cdot A_\mu + \frac{ig}{2} U \partial_\mu U^\dagger, \]  

(5.21)

which generalizes the infinitesimal transformation given in (5.16). Indeed if

\[ U = 1 + i g \delta \omega \cdot \frac{\tau}{2}, \]

(5.22)

we get

\[ \tau \cdot A_\mu U^\dagger = \tau \cdot A_\mu - \frac{\tau}{2} \cdot i g \delta \omega \cdot \partial_\mu \delta \omega \]

\[ = \tau \cdot A_\mu - \frac{\tau}{2} \cdot i g A_\mu \times \delta \omega + \frac{\tau}{2} \cdot \partial_\mu \delta \omega, \]  

(5.23)

which agrees with (5.16).

It is obviously convenient to define a matrix representation for the gauge fields:

\[ A_\mu = \frac{\tau}{2} \cdot A_\mu. \]  

(5.24)

Then the above gauge transformation (5.21) reads

\[ A_\mu U^\dagger = U A_\mu U^\dagger + \frac{ig}{2} U \partial_\mu U^\dagger, \]  

(5.25)

and the covariant derivative is

\[ D_\mu = \partial_\mu - ig A_\mu. \]  

(5.26)

This last includes the Abelian case, where \( U = e^{i \epsilon \lambda}, \tau/2 \rightarrow 1 \), which is a U(1) gauge group. To pick out the components of the gauge field, we recall that

\[ \tau_a \tau_b = \delta_{ab} + i \epsilon_{abc} \tau_c, \]  

(5.27a)
so because $\text{Tr} 1 = 2$ we have

$$\text{Tr} \tau_a \tau_b = 2 \delta_{ab}. \tag{5.27b}$$

Therefore

$$A_\mu^a = \text{Tr} \left( \tau_a \frac{T}{2} \cdot A \right) = \text{Tr} (\tau_a A_\mu). \tag{5.28}$$

Is the above interaction, minimal substitution, the end of the story? No, because we must consider the gauge field part of the action. Now

$$[D_\mu, D_\nu] = [\partial_\mu - igA_\mu, \partial_\nu - igA_\nu] = -ig(\partial_\mu A_\nu - \partial_\nu A_\mu) - g^2[A_\mu, A_\nu]. \tag{5.29}$$

Because our fields are non-Abelian, this last commutator is nonzero. Explicitly,

$$[A_\mu, A_\nu] = \left[ \frac{T}{2} \cdot A_\mu, \frac{T}{2} \cdot A_\nu \right] = \frac{i}{2} T \cdot (A_\mu \times A_\nu). \tag{5.30}$$

We will define the commutator as the field strength,

$$[D_\mu, D_\nu] = -igF_{\mu\nu}, \tag{5.31}$$

where

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) - ig[A_\mu, A_\nu]. \tag{5.32}$$

In terms of components, related to $F_{\mu\nu}$ by

$$F_{\mu\nu} = \frac{T}{2} \cdot F_{\mu\nu}, \tag{5.33}$$

we have

$$F_{\mu\nu}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + ge^{abc}A^b_\mu A^c_\nu, \tag{5.34a}$$

or

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + gA_\mu \times A_\nu. \tag{5.34b}$$

Why is this a useful quantity? Because it transforms covariantly,

$$F_{\mu\nu} \rightarrow F^U_{\mu\nu} = \frac{i}{g}[D^U_\mu, D^U_\nu] = UF_{\mu\nu}U^\dagger, \tag{5.35}$$

unlike the potential, as seen in (5.25). In infinitesimal form this means

$$F_{\mu\nu} \rightarrow F_{\mu\nu} - g\delta \omega \times F_{\mu\nu}. \tag{5.36}$$

From the field strength, the gauge field part of the Lagrangian can be constructed. (Why? Because it’s gauge invariant!)

$$\mathcal{L}_g = -\frac{1}{2} \text{Tr} F^{\mu\nu} F_{\mu\nu} = -\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu}. \tag{5.37}$$
Under a gauge transformation, \( \mathcal{L}_g \rightarrow \mathcal{L}_g' \). Explicitly,
\[
\mathcal{L}_g' = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu) \cdot (\partial^\mu A^\nu - \partial^\nu A^\mu) \\
- \frac{g}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu) \cdot (A^\mu \times A^\nu) \\
- \frac{g^2}{4} (A_\mu \cdot A_\nu A_\nu - A_\mu \cdot A_\nu A_\mu) \\
- g^2 \frac{1}{4} (A_\mu \cdot A_\nu A_\nu - A_\mu \cdot A_\mu A_\nu),
\]
(5.38)
because
\[
\epsilon_{abc} \epsilon^{ade} = \delta^{bd} \delta^{ce} - \delta^{de} \delta^{cd}.
\]
(5.39)
Note that the requirement of gauge invariance necessarily leads to cubic and quartic self-interactions of the gauge field, with the same coupling constant as appears in the gauge-field–fermion interaction.

### 5.1 Summary

For an arbitrary gauge group, the Lagrangian is
\[
\mathcal{L} = -\bar{\psi} \left( \frac{1}{i} \gamma_\mu D^\mu + m \right) \psi - \frac{1}{4} \text{Tr} F^2,
\]
(5.40)
where the gauge covariant derivative is
\[
D_\mu = \partial_\mu - ig A_\mu,
\]
(5.41)
and the gauge-covariant field strength is
\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu].
\]
(5.42)
This Lagrangian is invariant under the gauge transformations
\[
\psi \rightarrow U \psi, \\
A_\mu \rightarrow U A_\mu U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger, \\
F_{\mu\nu} \rightarrow U F_{\mu\nu} U^\dagger.
\]
(5.43a, 5.43b, 5.43c)

For SU(2), more explicitly, the Lagrangian is
\[
\mathcal{L} = -\bar{\psi} \left[ \gamma^\mu \frac{1}{i} \left( \partial_\mu - ig \frac{\tau}{2} \cdot A_\mu \right) + m \right] \psi - \frac{1}{4} F_{\mu\nu} \cdot F_{\mu\nu},
\]
(5.44)
where the field strength is
\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g A_\mu \times A_\nu.
\]
(5.45)
The gauge transformations are
\[
U = e^{ig \omega \cdot \tau / 2},
\]
(5.46)
so for an infinitesimal transformation, $\omega \rightarrow \delta \omega$,

$$\psi \rightarrow \left( 1 + ig\delta \omega \cdot \frac{\tau}{2} \right) \psi, \quad (5.47a)$$

$$A_\mu \rightarrow A_\mu - g\delta \omega \times A_\mu + \partial_\mu \delta \omega, \quad (5.47b)$$

$$F_{\mu\nu} \rightarrow F_{\mu\nu} - g\delta \omega \times F_{\mu\nu}. \quad (5.47c)$$

From the Lagrangian we can derive the equations of motion: Varying with respect to $\psi$ gives the gauge-covariant Dirac equation,

$$\left( \gamma^\mu \frac{1}{i} D_\mu + m \right) \psi = 0. \quad (5.48)$$

Under a $\delta A_\mu$ transformation,

$$\delta \mathcal{L} = \bar{\psi} \gamma^\mu g \delta A_\mu \psi - \text{Tr} F^{\mu\nu} (2\partial_\mu \delta A_\nu - 2igA_\mu \delta A_\nu - 2ig\delta A_\mu A_\nu), \quad (5.49)$$

so for SU(2) the change in the action is

$$\delta W = \int (dx) \delta A_\mu \left( \bar{\psi} \gamma^\mu g \frac{\tau}{2} \psi + \partial_\nu F^{\nu\mu} + gF^{\mu\nu} \times A_\nu \right), \quad (5.50)$$

where we have used (5.45). Thus, the Yang-Mills equation (the generalization of Maxwell’s equation) is

$$\partial_\nu F^{\mu\nu} = j_\mu, \quad (5.51)$$

with the current

$$j_\mu = \bar{\psi} \gamma^\mu g \frac{\tau}{2} \psi + gA_\nu \times F^{\nu\mu}. \quad (5.52)$$

The current has both fermion and gauge-boson pieces. Alternatively, we can define a gauge covariant derivative for the adjoint (isospin-1) representation of SU(2) by

$$\mathcal{D}_\nu = 1 \partial_\nu - gA_\nu \times, \quad (5.53a)$$

$$\mathcal{D}_\nu \cdot F^{\mu\nu} = gj_\mu = g\bar{\psi} \gamma^\mu \frac{\tau}{2} \psi, \quad (5.53b)$$

where $\mathcal{D}_\nu$ is a tensor, with components

$$(\mathcal{D}_\nu)_{ab} = \delta_{ab} \partial_\nu + g\epsilon_{abc} A_\nu^c. \quad (5.53c)$$