Stochastic Neural Networks with the Weighted Hebb Rule

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Abstract

Neural networks with synaptic weights constructed according to the weighted Hebb rule are studied in the presence of noise (finite temperature), when the number of stored patterns is finite. Although, for arbitrary weights not all of the stored patterns are *global* minima, there exists a temperature range in which only the stored patterns are minima of the free energy. In particular, a detailed analysis reveals that in the presence of a single extra pattern stored with an appropriate weight in the synaptic rule, the temperature at which the spurious minima of the free energy are eliminated is significantly lower than for a similar network without this extra pattern. The convergence time of the network, together with the overlaps of the equilibria of the network with the stored patterns, can thereby be improved considerably.

1 Introduction

The statistical mechanics of large neural networks with the Hebb rule prescription for the synaptic weights has been studied in detail and is now well–understood[1,2]. In this paper, we shall study the statistical mechanics of neural nets with synaptic weights which are constructed according to the weighted Hebb rule. For orthogonal patterns, the Hebb rule indeed stores the required patterns as fixed points of the deterministic updating dynamics, as is well known. The role of the weighted Hebb rule in the storage of non–orthogonal patterns was examined in ref.[3]. A Spin glass model for such neural networks was studied by L. Viana in [4], and at zero temperature the domains of attraction of the fixed points in the model were analyzed in [5,6]. All the above studies were concerned with the case where the number of stored patterns is finite; in the case where this number tends to infinity, a case of interest in a study of the effect of overloading on memory deterioration [7], extensive work has been carried out, as well [8].

Our principal motivation for returning to the weighted Hebb rule, in the case of a *finite* number of stored patterns, arises from the expectation that the presence of different weights for different patterns would affect the configuration of the free energy surface. There is the possibility that some of the degeneracy of the minima of the free energy would be lifted; in addition, the range of useful operating temperatures of the network would be changed. In [4], Viana considered the behaviour of the solutions as a function of temperature, but she did not derive the temperature range in which the spurious states are unstable. Furthermore, a large class of stable solutions, a subset of which was found at $T = 0$ in a study of the domains of attraction [5,6], was initially not found in the Mean-Field approximation [4].

In this article we re-examine the Mean-Field approximation of the model based on the weighted Hebb rule. We find solutions that were missed in earlier analyses, and shall find that in fact, the critical operating temperature of the network can be suitably lowered by a judicious choice of weights. The time needed for the network to converge to useful equilibrium states can be thereby reduced, since lower noise levels mean faster convergence times. Additionally, at lower temperatures, the overlaps of the network equilibria with the stored memories are larger; the overall quality of memory recall of the network can thus be significantly enhanced.

In the next section, we present the evaluation of the free energy along the lines of [1,4]. The stationary-point conditions yield the mean field equations (MFE's) for equilibrium states in the large–N limit. In section 3, we derive the stability conditions of the solutions to the mean field equations. In section 4, various critical temperatures for the existence of stable equilibria are calculated. In the last section we summarise our conclusions and contrast them with those of [4] and [5].

2 The weighted Hebb rule

We start with a network of N neurons with states $s_i(t) = \pm 1$ at time t. At time $t+1$, the probability of s_i flipping sign is

$$W(s_i \to -s_i) = (1 + \exp(2\beta s_i h_i))^{-1},$$

where

$$h_i = \sum_{j=1}^{N} J_{ij} s_j(t)$$

is the local field or potential at neuron i due to all the other neurons, and β is an inverse noise parameter(equivalently $T = 1/\beta$ is a 'temperature' parameter).

Since J_{ij} is symmetric, an energy function

$$H = -\frac{1}{2} \sum_{i \neq j} J_{ij} s_i s_j$$

can be attached to every configuration $[s]$. At zero temperature ($\beta \to \infty$), the network converges to a local minimum of this energy function.

Given a set of p patterns(finite in number) σ_i^μ, $\mu = 1, .., p$, to be stored, we could try to store these patterns by constructing the synaptic weights J_{ij} in the form

$$J_{ij} = \frac{1}{N} \sum_{\mu,\nu} g_{\mu\nu} \sigma_i^\mu \sigma_j^\nu$$

where the numbers $g_{\mu\nu}$ are positive. Without the $g_{\mu\nu}$ factor this would be nothing but the usual Hebb rule. However, with the $g_{\mu\nu}$, if we require the patterns σ to be stored as fixed points in the $\beta \to \infty$ limit, the statistically significant contributions come only from the $g_{\mu\mu}$ terms in J_{ij}. We shall therefore retain only these diagonal terms $g_{\mu\mu} \equiv g_\mu$ in the synaptic rule, yielding the weighted Hebb rule:

$$J_{ij} = \frac{1}{N} \sum_{\mu} g_\mu \sigma_i^\mu \sigma_j^\mu.$$

At finite temperatures T, one needs to look for minima of the free energy to identify metastable states of the network. Accordingly, we need to evaluate the partition function

$$Z = \mathrm{Tr} e^{-\beta H}.$$

Proceeding as in ref.[1], and assuming that the stored patterns are random, we define overlap variables $m_\mu \equiv \ll \sigma^\mu < s > \gg$. In the $N \to \infty$ limit, the free energy per neuron $f = F/N$ and the stationary point equations we get in the evaluation of Z then take the form

$$f(\beta) = \frac{1}{2} \sum_\mu^p \frac{1}{g_\mu} m_\mu^2 - \frac{1}{\beta} \ll \log(2\, \cosh \beta \vec{m}.\vec{\sigma}) \gg , \tag{1}$$

and

$$m_\mu = g_\mu \ll \sigma^\mu \tanh \beta \vec{m}.\vec{\sigma} \gg , \tag{2}$$

respectively[1].

We shall first look for solutions to the stationary point or mean field equations at zero temperature.

$\underline{T=0}$:

In this limit, the tanh function becomes a sign function, and $\log(2\cosh y) \to |y|$ as $|y| \to \infty$.

If $\vec{m}$ has only one non–zero component (the Mattis states), for instance $\vec{m} = (m, 0...0)$, then $m = g_1$, (up to an irrelevant sign), and $f = -(1/2) g_1$. From this we can see that the lowest-energy state, and hence a stable state, is

$$\vec{m} = \pm g_{max} (1, 0...0),$$

with g_{max} being the largest component of $\vec{g}$. The stability of these states can also be seen from the MFE and the free energy directly: we have $f = -(1/2) \sum (1/g_\mu) m_\mu^2$ and $\sum (1/g_\mu) m_\mu^2 \leq g_{max}$ (this follows from $\ll| m \cdot \sigma |\gg \leq [\sum (m^\mu)^2]^{1/2}$), implying that these Mattis states occupy global minima.

We note, however, that the other Mattis states, corresponding to $g < g_{max}$, are not *global* minima. Nevertheless, they are certainly local minima (and hence metastable states) at zero temperature, and they exist as stable states for sufficiently low temperatures as well, as we shall see.

[1] Our mean-field equation differs from that of ref. [4] only in convention; the order-parameter in the latter, written as q^μ, is related to ours as $q^\mu = m^\mu / g^\mu$.

For symmetric states with n non-zero components of the type $\vec{m} = m_n(1...1, 0...0)$, the MFE's imply that one must require,

$$g_1 = g_2 = ... = g_n \equiv g$$

in which case we get

$$m_n = \pm \frac{g}{n} \ll\mid z_n \mid\gg$$

and

$$f_n = -\frac{1}{2}\frac{g}{n} m_n^2 = -\frac{g}{2n} \ll\mid z_n \mid\gg^2,$$

where $z_n = \sum_{\mu=1}^{n} \sigma^\mu$. We shall call such states symmetric states corresponding to g.

These equations differ from those obtained in [1] only in the factor of g appearing on the right-hand side, resulting in the same ordering of the f_n's as that of [1].

We can also consider general states of the form $\vec{m} = (m_1, m_2, m_3, ..., m_n, 0, ..., 0)$, with distinct, nonzero m's. As can be seen from (2), the case of $n = 2$ is trivial, in that there are no nontrivial solutions with $m_1 \neq m_2$. To reduce technical complexity we restrict ourselves to the case of $n = 3$. As we shall see, since there are no nontrivial solutions with $n = 2$, this will be sufficient to establish a definitive conclusion. It is easy to see that the $T = 0$ limit of these states is $\vec{m} = (1/2)(g_1, g_2, g_3, 0, ..., 0)$, for *all* [2] positive g_μ. The stability of these solutions will be discussed in the next section.

$\underline{T \neq 0}$:

We shall assume that the states we wish to look at start appearing just below a temperature T(which depends on the state); correspondingly, the overlaps m are small near this temperature, and we can expand the cosh and tanh functions in a series in m, keeping only the first few terms. Then our equations become (to the appropriate order)

$$m_\mu = g_\mu \beta m_\mu (1 + \frac{2}{3}\beta^2 m_\mu^2 - \beta^2 \vec{m}^2) \tag{3}$$

[2] This can be seen as follows: The $T = 0$ limit of equation (2) is the same, but with the *tanh* replaced by the sign-function. Performing the quenched average leads to 3 equations - for $\mu = 1, 2, 3$. Then, one is faced with 4 separate cases to consider; $m_1 < m_2 < m_3$, $m_2 > m_1$ and $m_2 > m_3$, $m_2 < m_1$ and $m_2 < m_3$, and $m_1 > m_2 > m_3$, respectively. An exhaustive inspection, then, leads to the conclusion that $(1/2)(g_1, g_2, g_3, 0, ..., 0)$ is a solution if one has $g_1 + g_2 > g_3$ for the first, $g_1 + g_3 > g_2$ for the second, $g_1 + g_2 > g_3$ and $g_2 + g_3 > g_1$ for the third, and $g_2 + g_3 > g_1$ for the fourth case. This last condition alone is the one that was considered in [4]. However, we can see that all possible conditions on the 3 g's have appeared, and yet they correspond to the same solution, i.e. $(1/2)(g_1, g_2, g_3, 0, ..., 0)$. Hence, since any g has to satisfy one of these inequalities, this solution exists for all (positive) g_μ.

and

$$f(\beta) = \frac{1}{2}\sum_\mu \frac{1}{g_\mu} m_\mu^2 - \frac{1}{2}\beta \sum_\mu m_\mu^2 + \frac{1}{12}\beta^3 \ll (\vec{m}.\vec{\sigma})^4 \gg -T\log 2. \tag{4}$$

We can see that there exists a critical temperature, above which the only solution is the trivial one $\vec{m} = 0$. In particular, the critical temperature for the appearance of a Mattis state with one non-zero component m^μ is therefore $T_c^\mu = g_\mu$. For all of the Mattis states to exist as solutions of the stationary point equations, therefore, the operating temperature of the network must satisfy $T < g_s$ where g_s is the smallest of the weights g_μ.

The symmetric states $\vec{m} = m_n(1...1, 0, ...0)$, still require that we have $\vec{g} = (g, ...g, g_{n+1}, ..., g_p)$, with n g's. We find that for given g, the $n = 1$ state has the least free energy among the symmetric states corresponding to that g. In the next section we will see that these are in fact unstable above a certain critical temperature, and so we postpone the discussion of stability to that section.

For the general asymmetric states, having restricted our attention to the $n = 3$ case (i.e. $\vec{m} = (m_1, m_2, m_3, 0, ...0)$), we shall show that these states are also unstable at $T = g_\mu$. The stability of these states will be discussed at length in the next section.

3 Stability

The positivity of the eigenvalues of the stability matrix $\partial^2 f/\partial m_\mu \partial m_\nu$ assures the stability of the states. From (2) we get

$$\frac{\partial^2 f}{\partial m_\mu \partial m_\nu} = \frac{1}{g_\mu}\delta_{\mu\nu} - \beta(\delta_{\mu\nu} - Q_{\mu\nu}),$$

where

$$Q_{\mu\nu} = \ll \sigma^\mu \sigma^\nu \tanh^2 \beta \vec{m}.\vec{\sigma} \gg .$$

Zero temperature:

As we discussed in the previous section the Mattis states are stable at $T = 0$ for $\vec{g} = (g_{max}, g_2, ..., g_p)$, with $\vec{m} = (g_{max}, 0, ..., 0)$ being the global minimum.

For the symmetric states with n non-zero components $\vec{m} = m_n(1, 1, .., 1, 0, 0, ...0)$, with $\vec{g} = (g, ..., g, g_\alpha, ...)$, (where n of the g's are equal) and $\alpha = n+1, ..., p$, we find the eigenvalues of the stability matrix and find that, as in ref.[1], in the $T \to 0$ limit the eigenvalues, which depend on $q_n = Q_{\mu\mu}$ and $Q = Q_{\mu\nu}$ (with $\mu \neq \nu$), are all positive for the odd n states, while the even n states are all unstable due to the presence of negative eigenvalues.

In the case $n = 3$, for instance, and in the limit $T = 0$, we see that $q_n = 1$ and $Q = 0$, giving $\lambda_1 = 1/g$, $\lambda_2^\alpha = 1/g_\alpha$, and $\lambda_3 = 1/g$, all of which are positive, yielding stability. This is a point of departure from [4]; there, because of the condition $g_1 > g_2 > g_3$, it was impossible to allow for solutions of this type, and as a result, these solutions were not found. Although these solutions may not seem significantly different from the symmetric solutions found by Amit, et al. [1], as we shall see in the next section, the fact that we have not imposed any conditions on the g_μ will lead to a significant conclusion regarding the lowering of the critical temperatures of the network.

Similarly, for $T = 0$ and for asymmetric states, all the p eigenvalues reduce to their respective $1/g_\mu$, again yielding stability. This, too, was not considered in [4].

Finite temperatures:

For the symmetric states, at the temperature $T \sim g$, we have $q_n \approx \beta^2 m_n^2 n$, and $Q_n = (2q_n/n)$. Then we can see that

$$\lambda_3 = \frac{1}{g} - \beta(1 - q_n) - \beta Q_n \approx -\frac{4}{3n - 2}\frac{1}{g^2}(T - g)$$

which is clearly negative for $T \leq g$, except for $n = 1$ where $\lambda = \frac{2\beta}{g_\mu}(g_\mu - T) > 0$. The $(n > 1)$ symmetric states are therefore unstable at $T = g$.

Let us mention in passing that $\lambda_1 = \frac{1}{g} - \beta(1 - q_n) + \beta(n - 1)Q_n$ becomes, for $n = 1$ states, $+\frac{2(g-T)}{g^2}$, which is positive below the temperature $T = g$. The eigenvalue λ_3 is not present for $n = 1$ states. The sign of λ_2 depends explicitly on the various components of $\vec{g}$, and this shall be discussed further, below. However, that λ_3 is negative is sufficient to render the symmetric states with $n > 1$ unstable. The exact temperatures at which their stability, as well as the stability of the asymmetric states, is lost will be derived in the next section.

4 Critical Temperatures

The evaluation of critical temperatures requires the solution of the mean field equations in conjunction with various relations for the eigenvalues.

First, we deal with the symmetric states. We showed that of the eigenvalues (5) of the stability matrix, λ_1 is positive in the range $T = 0$ to $T \sim g$, λ_3 changes sign from $+$ to $-$, while the sign of λ_2 depends on the form of $\vec{g}$ explicitly (see below). Hence, there are two possibilities to consider: one is where λ_3 is set to zero, to find the critical temperature $T = T_c$ at which λ_3 changes sign, while λ_2 is constrained to be positive at that temperature T_c. The second case is where λ_2 is set to zero to find the critical temperature $T = T_c^*$, at which λ_2 changes sign, while λ_3 is constrained to be positive. The former gives

$$\beta m_3 = 0.94, \quad \frac{T_c}{g} = 0.46, \quad \text{for} \quad \frac{g_\alpha}{g} \leq 1.32, .$$

with the last constraint coming from the requirement $\lambda_2 \geq 0$ at $T = T_c$.

Our results up to this point do not differ significantly from those of [1]. However, let us go on further to the second case with $n = 1$.

Let g_s be the smallest of the g's, and consider the corresponding Mattis state $\vec{M} = (0, 0, .., m_s, 0, .., 0)$. The smallest of the eigenvalues in this case is λ_2^α with $g_\alpha = g_{max}$,

$$\lambda_2^\alpha = \frac{1}{g_{max}} - \beta(1 - q_1^s),$$

where g_{max} is the largest of the g's, and q_1^s is the corresponding value of q. Now to avoid spurious $n = 3$ states corresponding to g_{max} (which exist whenever g_{max} occurs at least three times in the set of g's), the operating temperature of the network must be greater than $T_c = 0.46 g_{max}$, as we have seen. At this temperature, in order for $\vec{M}$ to exist as a stable state, λ_2^α must be positive, or at best zero. This yields the constraint $g_s/g_{max} > 0.589$ on the value that the smallest g can take, if all the given patterns are to be stored as stable Mattis states of the network.

Turning now to the case of the $n = 3$ symmetric states corresponding to g, and for $g_\alpha/g > 1.32$, where g_α occurs only once or two times among the g's, we see that

$$\frac{T_c^*}{g} = \frac{g_\alpha}{g}(1 - q_n), \quad \text{with} \quad \frac{g_\alpha}{g} \geq 1.32$$

some of whose solutions can be tabulated as follows:

g_α/g	1.32	1.34	1.42	1.66	2.0	3.0
T_c^*/g	0.46	0.45	0.43	0.38	0.34	0.29

We can now see that, whereas for $g_\alpha \leq 1.32g$, the critical temperature is simply $0.46g$, for $g_\alpha \geq 1.32g$, the critical temperatures are all lower than the former. If

g_0 is the largest g which occurs at least three times, the operating temperature of the network must be at least $0.46 g_0$ if the largest g bigger than g_0, g_{max}, satisfies $g_{max} \leq 1.32 g_0$. This minimum necessary temperature for the avoidance of spurious equilibria is lowered when $g_{max} > 1.32 g_0$. In other words, by adding additional patterns with sufficiently large weights, we can lower the temperature above which there are no spurious states, leading to a "better" network. What is meant by "better" will be discussed in the next section.

The symmetric states with $n > 3$ can be shown to have even lower critical temperatures, exactly as in [1]. Therefore, it is sufficient to consider the $n = 3$ states only.

We now proceed to the case of the asymmetric states $\vec{m} = (m_1, m_2, m_3, 0, ..., 0)$ with a general weight vector, i.e. $\vec{g} = (g_1, g_2, g_3, g_\alpha)$. The MFE's can be written in terms of $x = \beta(m_1 + m_2 - m_3)$, $y = \beta(m_1 - m_2 + m_3)$, $z = \beta(-m_1 + m_2 + m_3)$, and the secular equation, dictating stability, can be written as

$$[\lambda^3 + \frac{1}{T} l_2 \lambda^2 + \frac{1}{T^2} l_1 \lambda + \frac{1}{T^3} l_0] \prod_{\alpha=n+1}^{p} [\frac{1}{g_\alpha} - \beta(1-q) - \lambda] = 0$$

where l_2, l_1, l_0 are all functions of x, y, $z, g_2/g_1$, g_3/g_1 and T/g_1. Here q is defined as

$$q \equiv \frac{1}{4}[\tanh^2(x+y+z) + \tanh^2 x + \tanh^2 y + \tanh^2 z].$$

Since we are generally interested in the temperature T_c at which a given eigenvalue becomes zero (i.e. changes sign from + to −), there are two separate cases we can consider: one is where $\lambda^{(\alpha)} = \frac{1}{g_\alpha} - \beta(1-q)$ is set to zero, while the other 3 eigenvalues (from the cubic part) are constrained to be nonnegative. The second choice is to set one of the 3 eigenvalues from the cubic part equal to zero and demand for $\lambda^{(\alpha)}$ and the remaining 2 eigenvalues to be non-negative.

The former case gives

$$(T_c^*/g_1) = \frac{g_\alpha}{g_1}(1-q) \tag{5}$$

and the positivity of the other eigenvalues can be insured by the constraints

$$l_2 < 0, \quad l_1 > 0, \quad \text{and} \quad l_0 < 0.$$

The first of these constraints, in conjunction with (5), simplifies to

$$\frac{g_\alpha}{g_1} > 3\left(1 + \frac{1}{(g_2/g_1)} + \frac{1}{(g_3/g_1)}\right)^{-1}$$

The last two constraints must be imposed numerically in finding T_c. Some results are shown in the table below for the case when $g_2 = g_3$. These spurious states are *stable* for $g_\alpha > g_\alpha^{min}$ and $T < T_c^*$.

g_2/g_1	0.6	0.8	0.95	1.0
g_α^{min}/g_1	2.0	1.89	1.6	1.32
T_c^*/g_1	0.18	0.27	0.37	0.46

In the second case, since we are interested only in the zero eigenvalues, it is sufficient to set $l_0 = 0$, and solve this equation numerically along with the MFE's. For the remaining eigenvalues to be nonnegative we must require

$$l_2 < 0, \quad l_1 > 0, \quad \text{and} \quad \frac{g_\alpha}{g_1} < \frac{(T/g_1)}{1-q}$$

Some results of this calculation are shown in the table below for $g_3 = g_2$.

g_2/g_1	1.1	1.2	1.3	1.4	1.5
T_c/g_1	0.29	0.22	0.19	0.15	0.11

Again, these spurious states are *stable* for $T < T_c$.

We note that for $\vec{g}$ of the form $(g_1, g_2, g_3, ...)$, with $g_1 = g_2 = 1$ and $g_3 = 1.32$, the critical temperature of the associated spurious state $(m_1, m_2, m_3, 0.., 0)$ with $m_1 = m_2$ is close to 0.19. If g_1 occurs at least three times in $\vec{g}$, the critical temperature of the $n = 3$ symmetric state corresponding to g_1 is 0.46. We can in fact make the general statement that if g_{max} is the largest component of $\vec{g}$, and g_0 the second largest, for $g_{max}/g_0 > 1.32$, the critical temperature above which there are *no* spurious states is determined by demanding the instability of spurious states with non-zero entries m_i of $\vec{m}$ corresponding to $g_i \leq g_0$.

We can present our results in the following format that clarifies the behaviour of T_c/g_1 for various values of g_2/g_1 and g_3/g_1:

$\frac{g_3}{g_1} \backslash \frac{g_2}{g_1}$	0.8	0.9	1.0	1.1	1.2
0.8	.12		.27		.15
0.9		.25	.33	.23	
1.0	.27	.33	.46	.29	.22
1.1		.23	.29	.37	
1.2	.15		.22		.35

The apparent symmetry of this table is simply due to the symmetry of the MFE's and the secular equation under the simultaneous exchange of $2 \leftrightarrow 3$ and $x \leftrightarrow y$. It is now evident that all the critical temperatures we have obtained for the asymmetric states are smaller than $0.46 g_0$ (where g_0 is the largest g that occurs at least three times), as one moves away from the Hebbian case at the center of the table.

5 Conclusion

Our investigation of the use of the weighted Hebb rule in Hopfield networks has revealed that the structure of the minima of the free energy at finite temperatures can be quite distinct from the case of the usual Hebb rule. In particular, by choosing the weighting factors for the various patterns appropriately, spurious states can be destabilised at a significantly lower temperature compared to that for the usual Hebb rule. When the operating temperature of the network is larger than the largest among the critical temperatures for the various spurious states, we have a network where only the Mattis states (corresponding to the stored patterns) are equilibria of the network.

Specifically, we can make the following rather general statements.

(1) If the largest of the g's, g_{max}, occurs at least three times or more, then the temperature range in which no spurious states exist is $0.46 g_{max} < T < g_{max}$. If the largest g which occurs at least three times is g_0, and the largest g, g_{max}, occurs no more than two times and satisfies $g_{max} > 1.32 g_0$, then the critical temperature above which no stable spurious states exist is smaller than $0.46 g_0$, and can be calculated as we have shown.

(2) If the smallest of the g's is g_{min}, and the largest one, g_{max}, occurs at least three times, and the constraint $g_{min}/g_{max} > 0.589$, is satisfied, all of the patterns to be stored exist as stable Mattis states in the range of temperatures where spurious states are excluded. If g_{max} occurs only once or twice, this constraint on the ratio of g_{min} to g_{max} is changed and can be calculated in a manner analogous to that shown in section 4.

In contrast to the results of ref.[4], we note that we have found stable spurious asymmetric states to exist at finite temperatures. Asymmetric solutions to the MFE's were also analyzed in [5] at zero temperature; an exhaustive finite temperature analysis of the mean field equations of the weighted Hebb rule Hopfield associative memory has, however, hitherto not appeared in the literature [9]. With the aim of improving the network performance, we have performed a detailed analysis of the stability conditions at finite temperatures, for completely arbitrary 'weights' g_μ. Our analysis has yielded various constraints that the 'weights' g corresponding to the different stored patterns must satisfy in order to completely avoid spurious attractors, and the corresponding range of useful operating temperatures.

One consequence of the lowering of the useful operating temperature is that convergence of the network to metastable states would be faster. A second consequence is that the overlaps of the equilibria of the network with the stored patterns would be larger due to the reduced temperature. Given a set of patterns to be stored, one could then simply put in an extra pattern weighted by a sufficiently larger weight g as compared to the g's of the other patterns to construct the synapses. The resulting network would then converge to an equilibrium state closer to one of the stored patterns and at a faster rate than a network constructed without this extra pattern being taken into account. Domains of attraction at zero temperature were estimated for the weighted Hebb rule network in [6]. It would be interesting to carry out detailed simulations of networks employing the weighted Hebb rule and to determine the relative sizes of the basins of attraction for the different stored patterns at finite temperatures as well.

Acknowledgements

C.M. is grateful to John Kuehler for assistance in the use of Fortran. We would also like to thank Jeff Arenzon for bringing ref. [4] to our attention.

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