Mechanics and Statistical Mechanics Qualifying Exam
Spring 2010

Problem 1: (10 Points)

A child of mass m is playing on a swing hanging from a support by a uniform chain of length L and negligible mass. In this question, you will explore the behavior of the swing in a number of situations.

a. Determine the equation of motion for the system in polar coordinates. **(2 Points)**

b. Consider small oscillations about the equilibrium position. What is the equation of motion for these conditions? **(1 Points)**

c. What is the oscillation frequency for the conditions described in part (b.)? **(2 Points)**

d. By starting at a sufficiently large speed at the bottom of the swing ($\theta = 0°$) the child can go 'over the top' ($\theta = 180°$). If the chain remains maximally extended at the top of the loop, what is the minimum velocity the child must have at the bottom of the loop ($\theta = 0$)? θ is the angle that the chain forms with the vertical. **(2 Points)**

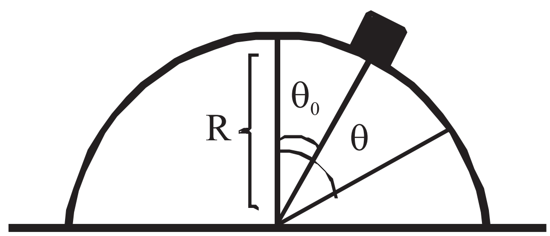
e. What is the minimum force applied to the child by the swing, that the child experiences at the bottom of the loop in part (d.)? **(1 Point)**

f. If the chain is replaced by a rigid rod of negligible mass, what is the minimum velocity of the child at the bottom required to go over the top? **(1 Point)**

g. What is the minimum force applied to the child by the swing, that the child experiences at the bottom of the loop in part (f.)? **(1 Point)**

Problem 2: (10 Points)

A hemisphere of radius R rests on the ground. A particle of mass m starts from rest on the sphere at an angle of θ_0 from the vertical that passes through the center of the sphere. Express answers in terms of R, θ_0 and the acceleration due to gravity near the surface of the earth, g.



a. The particle is released and slides without friction. At what angle, θ, measured relative to the vertical, does the particle leave the surface of the sphere? **(4 Points)**

b. What is the angle θ when $\theta_0 = 0$? **(1 Points)**

c. Assume the particle was released with $\theta_0 = 0$. Once the particle leaves the sphere, how long does it take it to hit the ground? **(3 Points)**

d. How far from the center of the sphere is the particle when it hits the ground? **(2 Points)**

Problem 3 (10 Points):

A nonrelativistic electron of mass m and charge $-e$ moves between a wire of radius a at negative electric potential $-\phi_0$ and a concentric cylindrical conductor of radius R at zero potential. There is a uniform constant magnetic field B parallel to the axis. The electric scalar and magnetic vector potentials can be written as:

$$\phi = -\phi_0 \frac{\ln(r/R)}{\ln(a/R)} \qquad\qquad \vec{\mathbf{A}} = \frac{1}{2} B r \hat{\theta}$$

where $\hat{\theta}$ is a unit vector in the increasing θ direction.

a. Give the Lagrangian and Hamiltonian in cylindrical coordinates. Specify all the constants of motion for this system justifying your answer. *Recall that the electric and magnetic potentials can be given in terms of a velocity dependent potential* $U(\vec{\mathbf{r}}, \dot{\vec{\mathbf{r}}}) = q\phi - q\vec{\mathbf{A}} \cdot \dot{\vec{\mathbf{r}}}$, *where* q *is the charge.* **(5 Points)**

b. For an electron starting at rest on the inner wire ($r = a$), there is a value of the magnetic field B_c such that for $B \le B_c$ the electron reaches the outer conductor and for $B > B_c$ it does not reach the outer conductor. Determine an expression for B_c in terms of the variables given; *you can assume that* $a \ll R$. **(5 Points)**

Problem 4 (10 Points):

Assume that air obeys the ideal gas equation. Take M to be the molar mass, P the pressure, R the ideal gas constant, T the temperature, z the altitude, ρ the density, and g the acceleration due to gravity.

a. The density of our atmosphere decreases with increasing altitude. This is a consequence of hydrostatic equilibrium, where the pressure of the air at an altitude z, must balance the pressure from below and the weight of the column of air above. Given that air has a mass density $\rho = MP/RT$, find dP/dz. Assume that the atmosphere is isothermal. Neglect the curvature of the earth and the variation of g with altitude. **(4 Points)**

b. Using the model in part (a.), consider a volume of air that is moved adiabatically within the atmosphere and able to do work on its surroundings; that is, expand and contract to maintain the same pressure as the surrounding air. If this section is moved upwards, it will cool as it is lifted, thus increasing in density compared to the surrounding air, and tend to sink back to its original altitude. Find dT/dz, the adiabatic lapse rate for the air. Assume the air is composed of diatomic molecules (N_2). (Hint: first find dT/dP). **(4 Points)**

The significance of the adiabatic lapse rate is that it determines the stability of the atmosphere to convection. The temperature in the lower part of the real atmosphere (troposphere) is not isothermal, but decreases with increasing altitude because it is heated by the ground. If the temperature gradient in the atmosphere is greater than the lapse rate, convection can occur.

c. If the section of air was wet so that condensation can occur, how does the lapse rate change? Explain your reasoning. **(1 Points)**

d. A helium balloon ascends in the atmosphere, expanding adiabatically just as the section of air in (b.). Will the lapse rate of helium be higher, the same, or lower than air? Explain. **(1 Points)**

Problem 5 (10 Points):

A system consists of N identical non-interacting particles in equilibrium with a heat bath. The total number of individual states available to each particle is $2\,N$. Of these states, N are degenerate with energy 0 and N are degenerate with energy ϵ. It is found by observation that the total energy of the system is $N\,\epsilon\,/3$.

a. What is the average number of particles in the excited state? **(1.5 Points)**

Find the temperature of the system under the following three different assumptions.

b. The particles are bosons. **(2 Points)**

c. The particles are fermions. **(2 Points)**

d. The particles obey a Boltzmann distribution. **(2 Points)**

e. Are the temperatures you found in (b.), (c.) and (d.) the same? Why or why not? Explain your answer. **(2.5 Points)**

Problem 6 (10 Points):

A large flat surface is in contact with a mono-atomic gas above it. The volume of gas above the surface acts as an infinite reservoir of gas atoms, but does not otherwise enter into the problem. The suface consist of a square lattice of sites that gas atoms can occupy; denote the number of gas atoms on site i by n_i, where $n_i \in \{0,1\}$, and the total number of lattice sites by N_s. The energy of the system is given by:

$$E(\{n_i\}) = -\left[\sum_i n_i\,\epsilon + v_0 \sum_i \sum_{j\in n.n.} n_i n_j\right] \tag{1}$$

where ϵ is a binding energy of atom to the substrate, v_0 is an interaction between adjacent atoms, and the sum over j is restricted to the nearest neighbors of i.

a. Write down an expression for the grand canonical partition function $Z(T,\mu)$. Your answer should be in the form of a sum over states. **(2 Points)**

b. Calculate the grand canonical free energy, $\Omega(T,\mu,N_s)$ when $v_0 = 0$. **(2 Points)**

c. Calculate N, the number of gas atoms adsorbed to the surface, as a function of T, μ and N_s when $v_0 = 0$. **(2 Points)**

d. When $v_0 \neq 0$ the problem is in general more difficult. To simplify it, replace n_j in the above sum by $\bar{n}$, a constant that will be set equal to the average occupation of any site. Calculate the number of gas atoms adsorbed to the surface, N, as a function of T, μ, N_s and $\bar{n}$. **(2 Points)**

e. Discuss the possibility of a phase transition in $\bar{n}$ as a function of β. This can be done by graphically investigating the requirement that $N(T,\mu,\bar{n})/N_s = \bar{n}$, or by returning to the expression for the energy given in equation (1) and mapping it on to other well known problems in statistical mechanics. **(2 Points)**