# Classical Mechanics and Statistical/Thermodynamics

**January 2009**

# Possibly Useful Information

Handy Integrals:

$$\begin{aligned}
\int_0^\infty x^n e^{-\alpha x}\,dx &= \frac{n!}{\alpha^{n+1}} \\
\int_0^\infty e^{-\alpha x^2}\,dx &= \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \\
\int_0^\infty x\ e^{-\alpha x^2}\,dx &= \frac{1}{2\alpha} \\
\int_0^\infty x^2\ e^{-\alpha x^2}\,dx &= \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}} \\
\int_{-\infty}^\infty e^{i\,a\,x - b\,x^2}\ dx &= \sqrt{\frac{\pi}{b}} e^{-a^2/4b}
\end{aligned}$$

Geometric Series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \equiv \zeta(p)$$

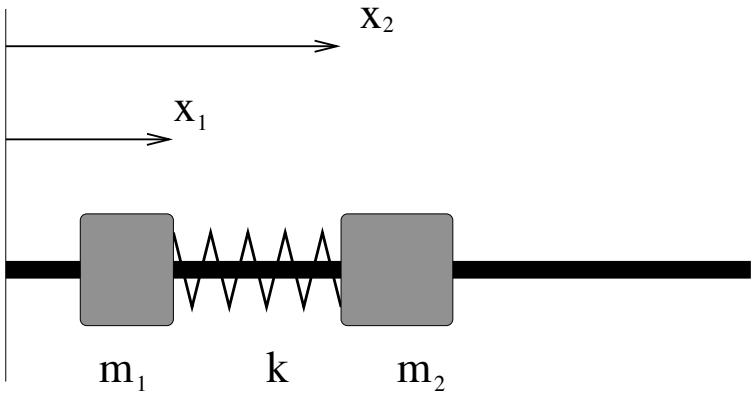
$$\sum_{n=1}^{\infty} \frac{z^p}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^{\infty} (-1)^p\ \frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

$$\begin{aligned}
\zeta(1) &= \infty & \zeta(-1) &= 0.0833333 \\
\zeta(2) &= 1.64493 & \zeta(-2) &= 0 \\
\zeta(3) &= 1.20206 & \zeta(-3) &= 0.0083333 \\
\zeta(4) &= 1.08232 & \zeta(-4) &= 0
\end{aligned}$$

# Classical Mechanics

1. Two masses, $m_1$ and $m_2$, are connected together by an ideal massless spring of spring constant $k$ and equilibrium length $l$, but are otherwise free to slide on a straight frictionless rail. Their positions with respect to a fixed origin are denoted $x_1$ and $x_2$ respectively.




   (a) Determine the equation of motion for each mass using Newton's Laws of Motion. (Do not solve them yet.) [2 pts.]

   (b) Write the Lagrangian for this system and use it to derive the equation of motion for each mass. (Again, do not solve them yet.) [3 pts.]

   (c) Using either of your results, determine the frequency of oscillation of the two masses about their center of mass. [2 pts.]

   (d) Given the initial conditions, $x_1(0) = 0$, $v_1(0) = 0$, $x_2(0) = l$ and $v_2(0) = v_0$, solve for the subsequent motion. [3 pts.]

2. A particle of mass $m$ moves under the influence of a central force whose potential is given by $V(r) = K\,r^3$, where $K > 0$.

   (a) For what energy and angular momentum will the orbit be a circle of radius $R_0$ about the origin? (3 pts.)

   (b) What is the period of this circular orbit? (2 pts.)

   (c) If the particle is slightly displaced from the circular orbit, what will be the period for small oscillations about $r = R_0$? (3 pts.)

   (d) For the $1/r$ gravitational potential we know that Kepler's Second Law: "A line joining a planet and the sun sweeps out equal areas during equal intervals of time." Does this hold true for the cubic potential as well? Prove your answer. (2 pts.)

3. You are in a rocket ship in outer space, initially at rest. You have a nuclear reactor that supplies a constant power, $P$, and a large supply of iron pellets. The iron pellets comprise 99/100 of your ship's mass, $m$. You can use the power to eject the tiny iron beads out the back of your ship with an electromagnetic "gun". You can control the *rate* at which you fire them and their velocity, but you are limited by your power plant. (You can't fire an arbitrarily large mass at an arbitrarily large velocity.) As you fire off the beads, your ship moves in the opposite direction to conserve momentum. In addition, the mass of your ship decreases.

   (a) Calculate your final speed as a functional of $m(t)$ and $\dot{m}(t) \equiv dm/dt$. Your expression should take the form of an integral over time, $0 < t < t_f$. (2 pts.)

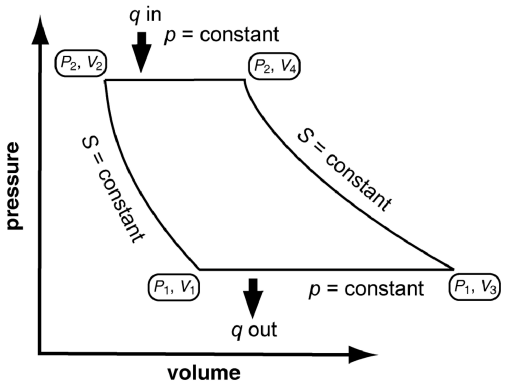
   (b) Find the function $m(t)$ that maximizes your final velocity after a time $t_f$. (4 pts.)

   (c) What is your final velocity? (2 pts.)

   (d) Prove that your answer in part (c) is larger than the velocity you would obtain by firing at a constant rate such that your pellets are used up by $t_f$. (2 pts.)

# Statistical Mechanics

4. The gas turbine (jet engine) can be modeled as a Brayton cycle. Below is the P-V diagram for this process.




Assume that the working fluid is an ideal monatomic gas.

(a) Calculate the work done by the gas on each step in the cycle. (3 pts.)

(b) Find the heat for each step in the cycle. (3 pts.)

(c) Find the efficiency of this engine. Your answer should be in terms of the pressures ($P_1$ and $P_2$) and the volumes ( $V_1$, $V_2$, $V_3$, and $V_4$). (3 pts.)

(d) To produce work, which way does the cycle operate? Clockwise or counter clockwise? (1 pt.)

5. By shining an intense laser beam on a semiconductor, one can create a metastable collection of electrons (charge $-e$ and effective mass $m_e$) and holes (charge $+e$ and effective mass $m_h$). These oppositely charged particles may pair up to form an *exciton*, or they may dissociate into a plasma. This problem considers a simple model of this process. In this problem the densities of electrons and holes are so low that you can ignore their fermionic nature and treat them as classical particles in three dimensions.

   (a) Calculate the free energy $F(T, V, N)$ of a gas of $N_e$ electrons and $N_h$ holes at temperature $T$, treating them as classical, non-interacting, ideal gas particles in a 3D volume $V$. (2 pts.)

   (b) By pairing into an exciton, each electron-hole pair lowers its energy by $\Delta E$. Calculate the free energy of a gas of $N_p$ excitons, treating them as classical, non-interacting, ideal gas particles. (2 pts.)

   (c) Calculate the chemical potentials $\mu_e$, $\mu_h$, and $\mu_p$ of the electrons, holes, and exciton pairs respectively. What is the condition of equilibrium between excitons and eletrons and holes? (3 pts.)

   (d) Consider the case where the numbers of electrons and holes are equal, so that $n_h = n_e \equiv n_0$. Determine the approximate density of excitons as a function of $n_0$ in the high temperature limit (when the exciton population is low). (3 pts.)

6. Consider a free, non-interacting spin zero Bose gas in two dimensions. The energy of each particle is given by:

$$\mathcal{E}(\vec{k}) = \hbar^2 k^2 / 2m$$

where $m$ is the mass of the boson. Assume your system is confined to a square region of length $L$ on a side.

(a) Write down an expression for the grand canonical free energy $\mathcal{G}(T, V, \mu)$ as a sum over $\vec{k}$ states. Do not evaluate the sum. (1 pt.)

(b) Calculate the number of particles in the system as a function of $T$, $V$ and $\mu$. (3 pts.)

(c) Analyze your expression for $N(T, V, \mu)$ in the limit $T \to 0$. What does it imply about the possibility of a Bose-Einstein transition in this system? (3 pts.)

(d) Prove that the pressure is equal to the energy density, so that $PV = U$. (Hint: you do not have to do any sums over states - you need only prove that this holds using analytic expressions for $P$ and $U$ in this particular system). (3 pts.)