PROBLEM 1

a. (3 pts) Using the luminosity equation for radiative transport

$$\frac{L}{4\pi r^2} = -\frac{4}{3}\frac{acT^3}{\kappa\rho}\frac{dT}{dr},$$

find an expression (using dimensional analysis) for the luminosity in terms of the mass such that L $\propto M^\alpha$. Assume that the opacity, κ, for this star is entirely due to electron scattering and is independent of mass or density.

b. (3 pts) Now assume that the actual luminosity of the star is

$$L = 4 \times 10^{33}(M/M_\odot)^\alpha ergs/sec$$

Using the α you found in part a), calculate the effective temperature of the star (assume a simple black body) that has M = 3M$_\odot$ and R = 2R$_\odot$.

For that temperature, use the Saha equation

$$log\Big[\frac{N_{i+1}}{N_i}\Big] = 2.5logT - \frac{5040}{T}\chi_i - 0.18$$

(where χ_i for H is 13.6 ev) to determine the **fraction** of total H atoms (in %) that are ionized in the atmosphere of this star. Do you expect that such a star will have strong or weak H spectral lines, and why? What is the spectral type of this star?

c. (4 pts) Calculate (in years) and compare the free-fall, Kelvin-Helmholtz and nuclear time scales for this same star.

The free-fall acceleration is given by

$$\frac{|d^2R|}{|dt^2|} = g.$$

Use dimensional analysis to get an expression for the free-fall time scale, τ_{f-f}, in terms of the average density $\bar{\rho}$. For our star of M = 3 M$_\odot$ and R = 2 R$_\odot$, calculate τ_{f-f}.

For the Kelvin-Helmholtz time scale, use the luminosity in b) and the total gravitational potential energy available to this 3M$_\odot$ star.

For the nuclear time scale for the same star assume that only 10% of the star's mass contributes to energy generation and the luminosity is given in b). [Assume .7 % mass loss in the nuclear conversion.] In terms of stellar evolution do your numbers for these time scales make sense, why or why not?

PROBLEM 2

a. (5 pts) H^- (binding energy $0.754\,\mathrm{eV}$) is an important source of opacity in the Sun. Using the Saha equation, calculate the ratio of the number of H^- ions to neutral hydrogen atoms in the Sun's photosphere. Take the temperature of the gas to be 5777K, and assume that the electron pressure is $1.5\,\mathrm{N\,m^{-2}}$. Note that the Pauli exclusion principle requires that only one state can exist for the ion because its two electrons must have opposite spins.

b. (5 pts) The Paschen series of hydrogen ($n = 3$) can contribute to the visible continuum for the Sun since the series limit occurs at 820.8 nm. However, it is the contribution from the H^- ion that dominates the formation of the continuum. Using the results of part a., along with the Boltzmann equation, estimate the number of H^- ions to hydrogen ions in the $n = 3$ state.

PROBLEM 3

A binary system consists of a neutron star (NS) with mass of 1.4 $M_\odot$, radius 1.00×10^6 cm and a small body of mass 1.00×10^{10} grams. The small body orbits the NS in an almost circular orbit with an initial period of 2.00×10^5 second.

a. (2 pts) What is the separation of the two bodies?

b. (2 pts) Give the total orbital energy of the system. You must include proper sign, proper number and proper unit.

c. (1 pt) The system is losing energy, say by gravitational radiation. This will cause the small body to spiral in and eventually collide with the neutron star. What is the orbital period of the system when the small body is orbiting just at the surface of the NS? (Ignore relativistic effects- the problem isn't meant to be that hard!)

d. (5 pts) Assume that, when the period is around 2.00×10^5 s, the orbital period is shrinking by 1.00×10^{-4} second each orbit as the system loses energy. What is the change of the separation of the bodies over one orbit as they inspiral? Hint: Differentiate Kepler's third law with respect to time to find the relationship between period change and separation change.

PROBLEM 4

a. (2 pts) Write or derive an equation for hydrostatic equilibrium in a form that is suitable for the interior of the sun, i.e., express dP/dr in terms of G, m, ρ, and r, where m is the mass interior to radius r and ρ is the mass density.

b. (1 pt) Rewrite the equation with m as the independent variable, i.e, dP/dm = ...

c. (1 pt) Use the dP/dm equation to obtain an approximate expression for the pressure at the center of the sun, in terms of G, M, and R, where M is the total mass of the sun and R is the solar radius.

d. (1 pt) To the nearest powers of ten, what are the temperature and the density at the center of the sun?

e. (1 pt) Write the "bottleneck" reaction (the least probable of the major reactions) for fusing hydrogen to helium in the core of the sun.

f. (2 pts) At the middle of the solar photosphere, where the optical depth at 5000 Å is about unity (1), what (to the nearest 1000 K) is the temperature? Is the mass density at this depth much greater than, much less than, or about equal to the density of air at sea level? Is hydrogen mostly ionized, mostly neutral, mostly locked up in diatomic molecules, or in some other form? What is the dominant source of opacity at 5000 Å? Identify the atomic process as specifically as you can.

g. (2 pts) In the approximation of local thermodynamic equilibrium (LTE), estimate the fraction of *all* hydrogen (ionized, neutral, molecular) that is in the Balmer (n = 2) level of neutral hydrogen.

PROBLEM 5

In the "Schuster-Schwarzschild Model" line formation is assumed to occur in a finite layer (the "reversing layer") of thickness τ_ν, above a sharp photosphere. The intensity from the photosphere is I_0. The incoming intensity at the surface is $I^- = 0$. In the reversing layer the continuum opacity is zero and the lines are purely scattering. Using the 2-Stream approximation,

$$I = \begin{cases} I^+ & : \quad \mu \geq 0 \\ I^- & : \quad \mu < 0 \end{cases}$$

Show that:

a. (2 pts)

$$\begin{aligned} H_\nu &= \frac{1}{4}(I^+ - I^-) \\ J_\nu &= \frac{1}{2}(I^+ + I^-) \end{aligned}$$

b. (1 pt) What is the value of ϵ?
Now we consider the transfer equation at only 2 angles with angles $\mu = \pm 1/2$ (this is a form of the discrete ordinates method).

c. (2 pts) Using the transfer equation at the 2 angles, show that $H_\nu =$ constant.

d. (3 pts) Using the transfer equation at the 2 angles, and the upper boundary condition show that $J(t_\nu)_\nu = 2H_\nu(2t_\nu + 1)$, where t_ν is a variable.

e. (2 pts) Using the lower boundary condition show that $H_\nu = \frac{I_0}{4(1+\tau_\nu)}$.

PROBLEM 6

The thermal history of the early universe can be described by the thermodynamics of a perfect (ideal) gas.

a. (4 pts) Derive the integral expression for the number density of particles at temperature T.

b. (2 pts) In the ultrarelativistic limit, derive the number density of bosons and fermions as functions of T.

c. (2 pts) How is the entropy density of the universe related to the number density of relativistic species of particles?

d. (2 pts) Derive the effective degeneracy factor g_* from the entropy density in terms of degeneracy factors of relativistic species of particles. (2 pts)

CONSTANTS

e = 4.8 x 10^{-10} esu
1 fermi = 10^{-13} cm
$L_\odot$ = 3.9 x 10^{33} ergs/sec
$M_\odot$ = 2 x 10^{33} gm
a = 7.56 x 10^{-15} erg $cm^{-3} deg^{-4}$
c= 3.0 x 10^{10} cm/sec
k = 1.38 x 10^{-16} erg/deg
$R_\odot$ = 7 x 10^{10} cm
1 year = 3.16 x 10^7 seconds,
N_A = 6.02 x 10^{23} moles/gm
G = 6.67 x $10^{-8} gm^{-1} cm^3 s^{-2}$
m_e=9.1 x 10^{-28} gm
h = 6.63 x 10^{-27} erg sec