Physics 5393
Problem Set 9–Due November 17, 2011

Problem (1)

In the Hilbert space, the eigenvectors of L^2 and L_z are denoted by $|\ell, m\rangle$. They obey the following eigenvalue equations

$$L^2|\ell, m\rangle = \ell(\ell+1)\hbar^2|\ell, m\rangle \quad \text{and} \quad L_3|\ell, m\rangle = m\hbar|\ell, m\rangle$$

and have additional interesting relations

$$L_\pm|\ell, m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m\pm 1)}|\ell, m\pm 1\rangle\,.$$

Calculate $\langle L_x\rangle$, $\langle L_y\rangle$, $\langle L_x^2\rangle$ and $\langle L_y^2\rangle$ in such a state denoted by $|\ell, m\rangle$.

Problem (2)

The Pauli spin matrices are defined as

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Three two-by-two matrices $(t_i, i = 1, 2, 3)$ satisfy the relations

$$[t_1, t_2] = -it_3, \quad [t_2, t_3] = -it_1, \quad [t_3, t_1] = -it_2.$$

Determine such a set of matrices.

(b) Show that

$$e^{i\pi\sigma_z/2} = i\sigma_z\,.$$

(c) Show that

$$U(\theta) = e^{-i\vec{\theta}\cdot\vec{\sigma}/2} = \cos\frac{\theta}{2} - i(\hat{\theta}\cdot\vec{\sigma})\sin\frac{\theta}{2}\,.$$

where $\vec{\theta} = \theta\hat{\theta}$ and $\hat{\theta} =$ the unit vector.

N.B. For a quantum operator, an expression of this form is to be interpreted as

$$e^{\Omega} = 1 + \Omega + \frac{1}{2}\Omega^2 + \frac{1}{3!}\Omega^3 + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}\Omega^n\,.$$

Problem (3)

Let us consider three 3×3 matrices G_i with elements given by

$$(G_i)_{jk} = -i\hbar\epsilon_{ijk} \quad i, j, k = 1, 2, 3$$

where j and k are the row and column indices.

(a) Show that G_i's satisfy the angular momentum commutation relations: $[G_i, G_j] = i\hbar\epsilon_{ijk}G_k$.

(b) Find the matrix G_3, then calculate the eigenvalues λ_i and normalized eigenvectors $|\lambda_i\rangle$ of G_3.

(c) Find a unitary matrix that transforms G_3 to J_3 for $j = 1$ with J_3 being diagonal

$$J_3 = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

N.B. This unitary matrix transforms the Cartesian space representation of the angular momentum operator G_i to its spherical basis representation J_i for $j = 1$. This problem is helpful in understanding the spin of photon.

Problem (4)

Let us consider a particle moving in a static and spherically symmetric potential with the wave function $\psi(r, \theta, \phi) \equiv \langle r, \theta, \phi|\psi\rangle$ in the spherical coordinate basis.

(a) Show that the operator

$$Q_r = i\frac{\partial}{\partial r}$$

is not Hermitian.

(b) Show that

$$P_r = -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)$$

is the correct definition of P_r in the sense that it will give the same expression as the radial part of the Laplacian in spherical coordinates and that it is Hermitian.

(c) With the replacement

$$P = -i\hbar\nabla$$

show that in the spherical coordinates

$$\frac{1}{r}(\vec{r} \cdot \vec{P}) = -i\hbar\frac{\partial}{\partial r}.$$

(d) In classical mechanics,

$$p_r = \frac{1}{r}(\vec{r} \cdot \vec{p}).$$

Find the correct P_r operator in quantum mechanics and show that it is the correct differential operator defined in part (b).