# PHYS 3803: Quantum Mechanics I
# **Problem Set 9–Due April 16, 2021**

Problem (1)

Let us define the wave function as

$$\psi_n(x) \equiv \langle x|n\rangle$$

for the $n-$th excited state of a quantum harmonic oscillator.

(a) In the $x-$basis, the ground state satisfies

$$a|0\rangle = 0\,, \quad \text{and} \quad \langle x|a|0\rangle = 0\,, \quad \text{with} \quad a = \sqrt{\frac{m\omega}{2\hbar}}\left(X + \frac{i}{m\omega}P\right).$$

Show that the ground state wave function is

$$\psi_0(x) = A_0\, e^{-\left(\frac{m\omega}{2\hbar}\right)x^2}$$

where $A_0$ is a normalization constant.

(b) Find the normalization constant $A_0$ in part (a).

Problem (2) [Griffiths 2.17]

In this problem we explore some useful theorems involving Hermite polynomials.

(a) The Rodrigues formula says that

$$H_n(\xi) = (-1)^n e^{\xi^2}\left(\frac{d}{d\xi}\right)^n e^{-\xi^2}\,.$$

Use it to derive $H_3$ and $H_4$.

(b) The following recursion relation gives you $H_{n+1}$ in terms of $H_n$ and $H_{n-1}$:

$$H_{n+1}(\xi) = 2\xi H_n(\xi) - 2nH_{n-1}(\xi)\,.$$

Use it, together with your answer in (a), to obtain $H_5$ and $H_6$.

(c) If you differentiate an $n$th-order polynomial, you get a polynomial of order $(n-1)$. For the Hermite polynomials, in fact,

$$\frac{dH_n}{d\xi} = 2nH_{n-1}(\xi)\,.$$

Check this by differentiating $H_5$ and $H_6$.

Problem (3)

Let's define the angular momentum operator as

$$\begin{aligned} L_i &= (\vec{X} \times \vec{P})_i = \epsilon_{ijk} X_j P_k, \quad i,j,k = 1,2,3 \text{ and} \\ \epsilon_{123} &= 1, \quad \epsilon_{213} = -1, \quad \epsilon_{iik} = 0, \end{aligned}$$

where $\epsilon_{ijk}$ is the anti-symmetric Levi-Civita symbol.

Find the following commutation relations:

(a) $[L_i, X_j]$,

(b) $[L_i, P_j]$,

(c) $[L_i, L_j]$,

(d) $[L_i, L^2]$,

where $L^2 = L_1^2 + L_1^2 + L_1^2 = L_j L_j$ (repeated indices imply summation).

To study the eigenvalue spectrum of the angular momentum operators, we define

$$L_+ \equiv L_1 + iL_2 \,, \quad \text{and} \quad L_- \equiv L_1 - iL_2 \,.$$

Find the following commutation relations:

(e) $[L_+, L_3]$,

(f) $[L_-, L_3]$,

(g) $[L_+, L_-]$.

Problem (4)

The Pauli spin matrices are defined as

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

Show that

(a) $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathrm{I} =$ the $2 \times 2$ unit matrix,

(b) $\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k, \quad i,j,k = 1,2,3$, where $\delta_{ij}$ is the Kronecker delta, and $\epsilon_{ijk}$ is the anti-symmetric Levi-Civita symbol,

(c) $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$ (the commutation relation),

(d) $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ (the anti-commutation relation).