PHYS 5213/4213: Nuclear and Particle Physics, Autumn 2021

Problem Set 9 – due November 18

Problems (1): Dirac Equation in an Electromagnetic Field

The Dirac equation for a fermion with charge Q in an electromagnetic field is

$$H\psi = E\psi \quad \text{with} \quad E = i\hbar\frac{\partial}{\partial t}$$

and the Hamiltonian is a 4×4 matrix of the following form

$$H = c\vec{\alpha} \cdot (\vec{P} - \frac{Q}{c}\vec{A}) + \beta mc^2 + Q\Phi\,.$$

In the non-relativistic limit, we have

$$E = E' + mc^2, \;\; \text{with} \;\; E' \ll mc^2\,.$$

Let us consider the wave function of the form

$$\psi(\vec{x}, t) = \psi'(\vec{x}, t)e^{-(i/\hbar)mc^2 t} \;\; \text{and} \;\; \psi' = \begin{pmatrix} \phi \\ \chi \end{pmatrix}\,.$$

Then the Dirac equation becomes

$$i\hbar\frac{\partial \phi}{\partial t} = \left[\frac{1}{2m}\left(\vec{P} - \frac{Q}{c}\vec{A}\right)^2 - \frac{Q}{mc}\vec{S} \cdot \vec{B} + Q\Phi\right]\phi$$

where

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}\,.$$

In addition, let us choose the Coulomb gauge $\nabla \cdot \vec{A} = 0$ and consider an electron in a system with

- a weak electric field, $|Q\Phi| \ll mc^2$, and
- a uniform weak magnetic field $\vec{B}$ with $\vec{A} = (1/2)\vec{B} \times \vec{r}$.

(a) Apply the tensor notation to show that

$$(\nabla \times \vec{A})_i = B_i\,.$$

(b) For a weak magnetic field, terms with $\vec{A}^2/c^2$ are negligible. Show that the equation of motion for ϕ becomes

$$i\hbar\frac{\partial\phi}{\partial t} = \left[\frac{\vec{P}^2}{2m} + \frac{e}{2mc}\vec{B}\cdot(\vec{L}+2\vec{S}) - e\Phi\right]\phi$$

where $Q = -e =$ charge of the electron, $\vec{S} =$ the spin angular momentum, and $\vec{L} =$ the orbital angular momentum.

Problem (2): Dirac Spinors

In natural units, the Lorentz covariant Dirac equation is

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0\,.$$

For a free particle, we have a four component eigenfunction of the following form

$$\psi(x) = u(p)e^{-ip\cdot x}$$

where u is a 4×1 spinor independent of x. That is

$$\psi_n = u_n e^{-ip\cdot x}, \quad n = 1,2,3,4\,.$$

That leads to

$$(\gamma^\mu p_\mu - m)u(p) = 0$$

or

$$(\not{p} - m)u(p) = 0$$

with the slash notation $\not{p} = \gamma^\mu p_\mu$.

We often choose

$$\begin{aligned} u^{(s)} &= N\begin{pmatrix}\phi^{(s)}\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\phi^{(s)}\end{pmatrix} \quad E>0\\ u^{(s+2)} &= N\begin{pmatrix}\frac{-\vec{\sigma}\cdot\vec{p}}{|E|+m}\chi^{(s)}\\ \chi^{(s)}\end{pmatrix} \quad E<0\end{aligned}$$

and

$$v^{(1,2)}(p) = u^{(4,3)}(-p)$$

where $s = 1,2$, $N = \sqrt{E+m}$ is the normalization constant, and

$$\phi^{(1)} = \chi^{(1)} = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \phi^{(2)} = \chi^{(2)} = \begin{pmatrix}0\\1\end{pmatrix}\,.$$

The spinors u (fermion) and v (anti-fermion) satisfy the following orthogonality relations

$$u^{(r)\dagger} u^{(s)} = 2E\delta_{rs} \quad \text{and} \quad v^{(r)\dagger} v^{(s)} = 2E\delta_{rs}\,.$$

Let us define $\bar{u} \equiv u^\dagger \gamma^0$ and $\bar{v} \equiv v^\dagger \gamma^0$.

(a) Show that $\bar{u}^{(s)} u^{(s)} = 2m$ and $\bar{v}^{(s)} v^{(s)} = -2m$.

(b) Derive the completeness relations

$$\begin{aligned}
\sum_{s=1,2} u^{(s)}(p)\bar{u}^{(s)}(p) &= \not{p} + m \\
\sum_{s=1,2} v^{(s)}(p)\bar{v}^{(s)}(p) &= \not{p} - m\,.
\end{aligned}$$

These relations are used extensively in the evaluation of Feynman diagrams.

Problem (3): Maxwell equations

In classical electrodynamics with the Heaviside-Lorentz conventions and $c = 1$, the 4-vector potential is defined as

$$A^\mu = (\Phi, \vec{A});$$

and the 4-vector current density is defined to be

$$J^\mu \equiv (\rho, \vec{J})\ .$$

Furthermore, let us defined a second-rank antisymmetric field-strength tensor as

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu,$$

then we can express the first two Maxwell equations with sources as

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu = J^\nu,$$

with the Lorenz condition $\partial_\mu A^\mu = 0$.

Exercise: Show that

$$\Box A^\nu = J^\nu,$$

that is

$$\begin{aligned}
\partial_\mu F^{\mu 0} &= \partial_\mu \partial^\mu A^0 = J^0 = \rho, \quad \text{and} \\
\partial_\mu F^{\mu i} &= \partial_\mu \partial^\mu A^i = J^i, \quad i = 1, 2, 3.
\end{aligned}$$

Problem (4): Polarization Vectors for a Massive Vector Particle

For a vector particle of mass M, energy E, and momentum $\vec{k}$ along the z axis, the polarization vectors can be expressed as

$$\begin{aligned} \epsilon^\mu(\lambda = \pm 1) &= \mp\frac{1}{\sqrt{2}}(0, 1, \pm i, 0), \quad \text{and} \\ \epsilon^\mu(\lambda = 0) &= (|\vec{k}|, 0, 0, E)/M \end{aligned}$$

that are called circular polarization vectors.

- Determine how $\vec{\epsilon}(\lambda = 1)$ and $\vec{\epsilon}(\lambda = -1)$ transform under a rotation ϕ about the z axis.

- Show that every polarization vector satisfies the following relations

$$\begin{aligned} \epsilon_\mu k^\mu &= 0 \quad \text{and} \\ \epsilon^*_\mu \epsilon^\mu &= -1 \,. \end{aligned}$$

- Show that the completeness relation is

$$\sum_\lambda \epsilon^*_\mu(\lambda)\epsilon_\nu(\lambda) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2} \,.$$

N.B. There is a minus sign in the definition of $\epsilon(\lambda = +1)$. This is a standard phase convention used in the construction of the spherical harmonics $Y_{\ell m}$ for $\ell = 1$ and $m = \pm 1$.