Problems (1): Dirac Equation in an Electromagnetic Field

The Dirac equation for a fermion with charge $Q$ in an electromagnetic field is

$$H\psi = E\psi \quad \text{with} \quad E = i\hbar \frac{\partial}{\partial t}$$

and the Hamiltonian is a $4 \times 4$ matrix of the following form

$$H = c\vec{\alpha} \cdot (\vec{P} - \frac{Q}{c} \vec{A}) + \beta mc^2 + Q\Phi.$$

In the non-relativistic limit, we have

$$E = E' + mc^2, \quad \text{with} \quad E' \ll mc^2.$$

Let us consider the wave function of the form

$$\psi(\vec{x}, t) = \psi'(\vec{x}, t)e^{-(i/\hbar)mc^2t} \quad \text{and} \quad \psi' = \begin{pmatrix} \phi \\ \chi \end{pmatrix}.$$

Then the Dirac equation becomes

$$i\hbar \frac{\partial \phi}{\partial t} = \left[ \frac{1}{2m} \left( \vec{P} - \frac{Q}{c} \vec{A} \right)^2 - \frac{Q}{mc} \vec{S} \cdot \vec{B} + Q\Phi \right] \phi$$

where

$$\vec{S} = \frac{\hbar}{2} \vec{\sigma}.$$

In addition, let us choose the Coulomb gauge $\nabla \cdot \vec{A} = 0$ and consider an electron in a system with

- a weak electric field, $|Q\Phi| \ll mc^2$, and
- a uniform weak magnetic field $\vec{B}$ with $\vec{A} = (1/2)\vec{B} \times \vec{r}$.

(a) Apply the tensor notation to show that

$$((\nabla \times \vec{A})_i = B_i.$$

(b) For a weak magnetic field, terms with $\vec{A}^2/c^2$ are negligible. Show that the equation of motion for $\phi$ becomes

$$i\hbar \frac{\partial \phi}{\partial t} = \left[ \vec{P}^2/2m + \frac{e}{2mc} \vec{B} \cdot (\vec{L} + 2\vec{S}) - e\Phi \right] \phi$$

where $Q = -e$ = charge of the electron, $\vec{S}$ = the spin angular momentum, and $\vec{L}$ = the orbital angular momentum.

**Problem (2): Dirac Spinors**

In natural units, the Lorentz covariant Dirac equation is

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0.$$  

For a free particle, we have a four component eigenfunction of the following form

$$\psi(x) = u(p)e^{-ip\cdot x}$$

where $u$ is a $4 \times 1$ spinor independent of $x$. That is

$$\psi_n = u_ne^{-ip\cdot x}, \ n = 1, 2, 3, 4.$$  

That leads to

$$(\gamma^\mu p_\mu - m)u(p) = 0$$

or

$$(\not{p} - m)u(p) = 0$$

with the slash notation $\not{p} = \gamma^\mu p_\mu$.

We often choose

$$u^{(s)} = N \left( \frac{\phi^{(s)}}{\sqrt{E+m}} \right) \quad E > 0$$

$$u^{(s+2)} = N \left( \frac{-\vec{\sigma} \cdot \vec{p}}{|E|+m} \chi^{(s)} \right) \quad E < 0$$

and

$$v^{(1,2)}(p) = u^{(4,3)}(-p)$$

where $s = 1, 2$, $N = \sqrt{E+m}$ is the normalization constant, and

$$\phi^{(1)} = \chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \phi^{(2)} = \chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$
The spinors $u$ (fermion) and $v$ (anti-fermion) satisfy the following orthogonality relations

$$u^{(r)\dagger}u^{(s)} = 2E\delta_{rs} \quad \text{and} \quad v^{(r)\dagger}v^{(s)} = 2E\delta_{rs}.$$ 

Let us define $\bar{u} \equiv u^\dagger\gamma^0$ and $\bar{v} \equiv v^\dagger\gamma^0$.

(a) Show that $\bar{u}^{(s)}u^{(s)} = 2m$ and $\bar{v}^{(s)}v^{(s)} = -2m$.

(b) Derive the completeness relations

$$\sum_{s=1,2} u^{(s)}(p)\bar{u}^{(s)}(p) = \not{p} + m$$

$$\sum_{s=1,2} v^{(s)}(p)\bar{v}^{(s)}(p) = \not{p} - m.$$ 

These relations are used extensively in the evaluation of Feynman diagrams.

**Problem (3): Maxwell equations**

In classical electrodynamics with the Heaviside-Lorentz conventions and $c = 1$, the 4-vector potential is defined as

$$A^\mu = (\Phi, \vec{A});$$

and the 4-vector current density is defined to be

$$J^\mu \equiv (\rho, \vec{J}).$$

Furthermore, let us defined a second-rank antisymmetric field-strength tensor as

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu,$$

then we can express the first two Maxwell equations with sources as

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu = J^\nu,$$

with the Lorenz condition $\partial_\mu A^\mu = 0$.

**Exercise:** Show that

$$\Box A^\nu = J^\nu,$$

that is

$$\partial_\mu F^{\mu0} = \partial_\mu \partial^\mu A^0 = J^0 = \rho, \quad \text{and}$$

$$\partial_\mu F^{\mu i} = \partial_\mu \partial^\mu A^i = J^i, \quad i = 1, 2, 3.$$
Problem (4): Polarization Vectors for a Massive Vector Particle

For a vector particle of mass $M$, energy $E$, and momentum $\vec{k}$ along the $z$ axis, the polarization vectors can be expressed as

$$
\epsilon^\mu(\lambda = \pm 1) = \mp \frac{1}{\sqrt{2}}(0, 1, \pm i, 0), \quad \text{and}
$$

$$
\epsilon^\mu(\lambda = 0) = (|\vec{k}|, 0, 0, E)/M
$$

that are called circular polarization vectors.

- Determine how $\vec{\epsilon}(\lambda = 1)$ and $\vec{\epsilon}(\lambda = -1)$ transform under a rotation $\phi$ about the $z$ axis.

- Show that every polarization vector satisfies the following relations

$$
\epsilon_\mu k^\mu = 0 \quad \text{and} \quad \epsilon^*_\mu \epsilon^\mu = -1.
$$

- Show that the completeness relation is

$$
\sum_\lambda \epsilon^*_\mu(\lambda)\epsilon_\nu(\lambda) = -g_{\mu\nu} + \frac{k^\mu k_\nu}{M^2}.
$$

N.B. There is a minus sign in the definition of $\epsilon(\lambda = +1)$. This is a standard phase convention used in the construction of the spherical harmonics $Y_{\ell m}$ for $\ell = 1$ and $m = \pm 1$. 