Physics 5393
Problem Set 8–Due November 1, 2011

Problem (1)

In a system with a harmonic oscillator, a particle starts out in

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} (\, |0\rangle + |1\rangle \,) \, . \tag{1}$$

at $t = 0$ with $E_n = \hbar\omega(n + 1/2)$.

(a) Find $|\psi(t)\rangle$ in terms of ω.

(b) Find $\langle X(0)\rangle$, $\langle P(0)\rangle$, $\langle X(t)\rangle$, and $\langle P(t)\rangle$.

(c) Find $\frac{d}{dt}\langle X(t)\rangle$ and $\frac{d}{dt}\langle P(t)\rangle$ by using the Ehrenfest's Theorem and solve for $\langle X(t)\rangle$, and $\langle P(t)\rangle$.

Problem (2)

Let's define the angular momentum operator as

$$\begin{aligned} L_i &= (\vec{X} \times \vec{P})_i = \epsilon_{ijk} X_j P_k, \quad i,j,k = 1,2,3 \text{ and} \\ \epsilon_{123} &= 1, \quad \epsilon_{213} = -1, \quad \epsilon_{iik} = 0, \end{aligned}$$

where ϵ_{ijk} is the anti-symmetric Levi-Civita symbol.

Find the following commutation relations:

(a) $[L_i, X_j]$,

(b) $[L_i, P_j]$,

(c) $[L_i, L_j]$,

(d) $[L_i, L^2]$,

where $L^2 = L_1^2 + L_1^2 + L_1^2 = L_j L_j$ (repeated indices imply summation).

Problem (3)

To study the eigenvalue spectrum of the angular momentum operators, we define

$$\begin{aligned} L_+ &\equiv L_1 + iL_2 \ , \quad \text{and} \\ L_- &\equiv L_1 - iL_2 \ . \end{aligned}$$

Find the following commutation relations:

(a) $[L_+, L_3]$,

(b) $[L_-, L_3]$,

(c) $[L_+, L_-]$.

Problem (4)

The Pauli spin matrices are defined as

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

Show that

(a) $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathrm{I} =$ the 2×2 unit matrix,

(b) $\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k, \quad i, j, k = 1, 2, 3$, where δ_{ij} is the Kronecker delta, and ϵ_{ijk} is the anti-symmetric Levi-Civita symbol,

(c) $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$ (the commutation relation),

(d) $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ (the anti-commutation relation).