Problem (1)

In a system with a harmonic oscillator, a particle starts out in

\[ |\psi(0)\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle). \tag{1} \]

at \( t = 0 \) with \( E_n = \hbar \omega (n + 1/2) \).

(a) Find \( |\psi(t)\rangle \) in terms of \( \omega \).

(b) Find \( \langle X(0) \rangle, \langle P(0) \rangle, \langle X(t) \rangle, \) and \( \langle P(t) \rangle \).

(c) Find \( \frac{d}{dt}\langle X(t) \rangle \) and \( \frac{d}{dt}\langle P(t) \rangle \) by using the Ehrenfest’s Theorem and solve for \( \langle X(t) \rangle \), and \( \langle P(t) \rangle \).

Problem (2)

Let’s define the angular momentum operator as

\[ L_i = (\vec{X} \times \vec{P})_i = \epsilon_{ijk} X_j P_k, \quad i, j, k = 1, 2, 3 \quad \text{and} \]

where \( \epsilon_{ijk} \) is the anti-symmetric Levi-Civita symbol.

Find the following commutation relations:

(a) \( [L_i, X_j], \)

(b) \( [L_i, P_j], \)

(c) \( [L_i, L_j], \)

(d) \( [L_i, L^2], \)

where \( L^2 = L_1^2 + L_2^2 + L_3^2 = L_j L_j \) (repeated indices imply summation).

Problem (3)

To study the eigenvalue spectrum of the angular momentum operators, we define

\[ L_+ \equiv L_1 + iL_2, \quad \text{and} \]

\[ L_- \equiv L_1 - iL_2. \]

Find the following commutation relations:
(a) \([L_+, L_3]\),

(b) \([L_-, L_3]\),

(c) \([L_+, L_-]\).

**Problem (4)**

The Pauli spin matrices are defined as

\[
\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (2)

Show that

(a) \(\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I = \) the 2 \(\times\) 2 unit matrix,

(b) \(\sigma_i \sigma_j = \delta_{ij} + i \epsilon_{ijk} \sigma_k, \quad i, j, k = 1, 2, 3,\) where \(\delta_{ij}\) is the Kronecker delta, and \(\epsilon_{ijk}\) is the anti-symmetric Levi-Civita symbol,

(c) \([\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k\) (the commutation relation),

(d) \(\{\sigma_i, \sigma_j\} = 2\delta_{ij}\) (the anti-commutation relation).