PHYS 3803: Quantum Mechanics I
Problem Set 8–Due April 07, 2021

Problem (1) [Griffiths 2.22] **The Gaussian Wave Pocket**

A free particle has the initial wave function

$$\psi(x) = \Psi(x,0) = A\,e^{-ax^2}\,, \qquad a \in \mathcal{R} \quad \text{and} \quad a > 0\,,$$

where A and a are constants.

(a) Normalize $\psi(x)$.

(b) Find $\phi(k)$ as the Fourier transform of $\psi(x)$, then find the wave function $\Psi(x,t)$ as

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i[kx-(\hbar k^2/2m)t]}\,\phi(k)\,dk\,.$$

(c) Find $|\Psi(x,t)|^2$. Express your answer in terms of the quantity

$$w = \sqrt{\frac{a}{1+(2\hbar a t/m)^2}}\,.$$

Sketch $|\Psi|^2$ as a function of x at $t=0$, and again for some very large t.

Problem (2) [Griffiths 2.33]

A particle moves in a system with a **rectangular barrier** of width $2a$ that has the potential energy

$$V(x) = \begin{cases} V_0, & \text{for } -a \le x \le a \text{ with } a > 0, \text{ and} \\ 0, & \text{for } |x| > a, \end{cases}$$

and V_0 is larger than the total energy ($V_0 > E > 0$). Let us define

$$\begin{aligned} k &= \frac{\sqrt{2mE}}{\hbar}\,, \qquad \text{and} \\ \lambda &= \frac{\sqrt{2m(V_0-E)}}{\hbar}\,. \end{aligned}$$

(a) Show that the wave function in Region I is

$$\psi_1(x) = Ae^{ikx} + Be^{-ikx}\,, \quad \text{for} \quad x < -a,$$

and the wave function in Region III is

$$\psi_3(x) = Fe^{ikx}\,, \quad \text{for} \quad x > a,$$

in the case that there is no incoming wave in Regin III.

(b) Show that the wave function in Region II is

$$\psi_2(x) = Ce^{\lambda x} + De^{-\lambda x}$$

for $-a \leq x \leq a$.

(c) Apply boundary conditions and determine the transmission coefficient

$$T \equiv \left|\frac{F}{A}\right|^2$$

such that

$$\begin{aligned} T^{-1} &= \left|\frac{A}{F}\right|^2 \\ &= 1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2\left(2a\sqrt{2m(V_0 - E)}\right) \\ &= 1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2(2a\lambda) \end{aligned}$$

for $V_0 > E > 0$ (quantum tunneling).

Hint: $\cosh^2 x - \sinh^2 = 1$.

Problem (3) [Griffiths 2.49]

A simple harmonic oscillator has the Hamiltonian expressed as

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2 .$$

(a) Show that

$$\Psi(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{\alpha^2}{2}(1 + e^{-2i\omega t}) + \frac{i\hbar t}{m} - 2\alpha x e^{-i\omega t}\right)\right]$$

satisfies the Schrödinger equation for the harmonic oscillator. Here α is a real constant with the dimensions of length.

(b) Compute $\langle X \rangle$ and $\langle P \rangle$ and check that Ehrenfest's theorem is satisfied.

Problem (4)

Let us consider a quantum harmonic oscillator.

(a) Find $\langle X \rangle, \langle P \rangle, \langle X^2 \rangle, \langle P^2 \rangle$ and $\Delta X \Delta P$ in the state $|n\rangle$.

(b) What is the uncertainty relation for the ground state.