PHYS 5213/4213: Nuclear and Particle Physics, Autumn 2021

Problem Set 8 – due November 11

Problem 1: Dirac Matrices

To combine quantum mechanics with special relativity, Dirac suggested a linear relationship between the Hamiltonian (H) and the momentum ($\vec{P}$)

$$H = c\vec{\alpha} \cdot \vec{P} + \beta mc^2$$

where α_i and β are $N \times N$ matrices.

If H is the correct Hamiltonian, by squaring it we should get back the relation from relativity. That is

$$\begin{aligned} c^2\vec{P}^2 + m^2c^4 &= H^2 \\ &= c^2\left[\frac{1}{2}\{\alpha_i, \alpha_j\}P_iP_j + \{\alpha_i, \beta\}mcP_i + \beta^2m^2c^2\right] . \end{aligned}$$

Thus it is clear that the left hand side equals the right hand side if

$$\begin{aligned} \beta^2 &= I \\ \alpha_i^2 &= I \end{aligned}$$

and

$$\begin{aligned} \{\alpha_i, \beta\} &= 0 \\ \{\alpha_i, \alpha_j\} &= 0 \quad \text{for } i \neq j \end{aligned}$$

or

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} .$$

For $N = 4$ let us choose β to be diagonal and traceless.

$$\begin{aligned} \beta &= \begin{pmatrix} I & O \\ O & -I \end{pmatrix} , \\ I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \\ O &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} . \end{aligned}$$

In addition, let's choose

$$\alpha_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix},$$

where A_i, B_i, C_i and D_i are 2×2 matrices.

Apply $\{\alpha_i, \beta\} = 0$, $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$, $\alpha_i^\dagger = \alpha_i$, and show that

$$\text{(a)}\ \ \alpha_i = \begin{pmatrix} 0 & B_i \\ B_i^\dagger & 0 \end{pmatrix} ,$$

and

$$\text{(b)}\ \ Bi^\dagger B_j + B_j^\dagger B_i = 0 \ .$$

We can choose $B_i = \sigma_i$ with

$$\begin{aligned} \sigma_i^\dagger &= \sigma_i \ , \ \text{ and} \\ \sigma_i\sigma_j + \sigma_j\sigma_i &= 2\delta_{ij} \ , \end{aligned}$$

then α_i's become

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} .$$

Problem 2: Spin of the Electron

Let us define a generalized Pauli matrix (spin operator)

$$\Sigma_i \equiv \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \ \text{ with } \ \xi \equiv \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

such that

$$\alpha_i = \xi\Sigma_i = \Sigma_i\xi \ .$$

Thus we see that ξ commutes with Σ_i,

$$[\Sigma_i, \Sigma_j] = 2i\epsilon_{ijk}\Sigma_k \ \text{ and } \ [\Sigma_i, \beta] = 0 \ .$$

In terms of these matrices, we can write down the Dirac Hamiltonian as

$$H = c\vec{\alpha} \cdot \vec{P} + \beta mc^2 + V(r) = c\xi\Sigma_i \cdot P_i + \beta mc^2 + V(r) \ .$$

Here we have added a spherically symmetric potential to allow for interaction. We know that since the Hamiltonian is invariant under rotations, angular momentum must be conserved.

(a) Show that

$$[L_3, H] = i\hbar c\xi(\Sigma_1 P_2 - \Sigma_2 P_1) \ .$$

(b) Show that

$$[\Sigma_3, H] = 2ic\xi(\Sigma_2 P_1 - \Sigma_1 P_2)\ .$$

(c) Find the operator (J_3) for the total angular momentum that commutes with the Hamiltonian as a function of L_3 and Σ_3.

(d) Find the spin operator ($\vec{S} = \vec{J} - \vec{L}$) and the spin ($s$) for the electron.

Problem 3: Dirac Equation in an Electromagnetic Field

The classical interaction of a particle with charge Q can be expressed as

$$U(\vec{x}, \dot{\vec{x}}) = -\frac{Q}{c}\vec{v}\cdot\vec{A} + Q\Phi$$

from a 4-vector potential $A^\mu = (\Phi, \vec{A})$ where $\Phi =$ the electric potential and $\vec{A} =$ the magnetic vector potential. We can define a velocity operator $\vec{v} = c\vec{\alpha}$ in quantum mechanics for the Dirac equation. The the Hamiltonian becomes

$$H = c\vec{\alpha}\cdot(\vec{P} - \frac{Q}{c}\vec{A}) + \beta mc^2 + Q\Phi$$

and the Dirac equation becomes

$$[c\vec{\alpha}\cdot(\vec{P} - \frac{Q}{c}\vec{A}) + \beta mc^2 + Q\Phi]\psi = i\hbar\frac{\partial\psi}{\partial t}$$

In the non-relativistic limit, we have

$$E = E' + mc^2, \ \text{ with } \ E' \ll mc^2\ .$$

Let us consider the wave function of the form

$$\psi(\vec{x}, t) = \psi'(\vec{x}, t)e^{-(i/\hbar)mc^2 t} \ \text{ and } \ \psi' = \begin{pmatrix} \phi \\ \chi \end{pmatrix}\ .$$

Show that

$$i\hbar\frac{\partial\phi}{\partial t} = \left[\frac{1}{2m}\left(\vec{P} - \frac{Q}{c}\vec{A}\right)^2 - \frac{Q}{mc}\vec{S}\cdot\vec{B} + Q\Phi\right]\phi$$

where

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}\ .$$

Hint: Let's define

$$\vec{\pi} = \vec{P} - \frac{Q}{c}\vec{A}$$

and rewrite the Dirac equation as

$$i\hbar\frac{\partial}{\partial t}\psi' = \left[c\vec{\alpha}\cdot\vec{\pi} + (\beta - I)mc^2 + Q\Phi\right]\psi'$$

with

$$(-2mc^2 + Q\Phi - i\hbar\frac{\partial}{\partial t})\chi \simeq -2mc^2\chi \; .$$