Physics 5393
Problem Set 7–Due October 20, 2011

Problem (1)

In the energy basis of a harmonic oscillator, we have

$$\begin{aligned} X &= \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \quad \text{and} \\ P &= -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger) \end{aligned}$$

where a and $a^\dagger$ are the lowering and raising operators, respectively.

Calculate the following matrix elements for a quantum oscillator

(a) $\langle 2|X^3|0\rangle$, and

(b) $\langle 2|X^3|1\rangle$.

Problem (2)

Apply the generating function for the Hermite polynomial

$$G(\xi, s) = e^{\xi^2 - (s-\xi)^2} = \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(\xi)$$

and show that

$$\begin{aligned} &\text{(a) } H_n(\xi) = \frac{\partial^n G(\xi, s)}{\partial s^n}|_{s=0}; \\ &\text{(b) } \frac{\partial^n G(\xi, s)}{\partial s^n}|_{s=0} = (-1)^n e^{\xi^2} \frac{\partial^n}{\partial \xi^n} e^{-\xi^2} \ = \ H_n(\xi); \ \text{ and} \\ &\text{(c) } \int_{-\infty}^{\infty} d\xi H_n(\xi) H_m(\xi) e^{-\xi^2} = \begin{cases} 0 & \text{if } n \neq m, \text{ and} \\ \sqrt{\pi} 2^n \cdot n! & \text{for } n = m. \end{cases} \end{aligned}$$

(d) Find the normalized wave function $u_n(x)$ with

$$u_n(\xi) = A_n H_n(\xi) e^{-\frac{1}{2}\xi^2},$$

where A_n is real, $\xi = \alpha x$, $\alpha = \sqrt{m\omega/\hbar}$, and $u_n(x)$ is an eigenfunction for the one-dimensional Simple Harmonic Oscillator, corresponding to the energy $(n + 1/2)\hbar\omega$.

Problem (3)

A simple harmonic oscillator has the Hamiltonian expressed as

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2 \, .$$

(a) Show that

$$\Psi(x,t) \;=\; \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{\alpha^2}{2}(1+e^{-2i\omega t}) + \frac{i\hbar t}{m} - 2\alpha x e^{-i\omega t}\right)\right]$$

satisfies the Schrödinger equation for the harmonic oscillator. Here α is a real constant with the dimensions of length.

(b) Compute $\langle X \rangle$ and $\langle P \rangle$ and check that Ehrenfest's theorem is satisfied.

Problem (4)

A particle moves in three dimensions in a potential of the form

$$V(\vec{r}) = \begin{cases} \frac{1}{2}m\omega^2(x^2+y^2), & \text{for } -a \le z \le a \text{ with } a > 0, \text{ and} \\ \infty, & \text{for } z^2 > a^2, \end{cases}$$

where m and ω are constants.

Let us consider the wave function of this system as

$$\psi_{l,m,n}(\vec{r}) = \psi_a^l(x)\psi_b^m(y)\psi_c^n(z).$$

Find formulas for its energy eigenvalues, the degeneracy of each energy level, and the eigenfunctions associated with them.