Problem (1)

The magnitude of the magnetic dipole moment (MDM) of an elementary fermion such as a quark of charge $q$, mass $m$, and spin $s$ is

$$\vec{\mu} = \left(\frac{q\hbar}{mc}\right)\vec{s}. $$

(1)

(a) What are the values of the MDM of the up quark ($\mu_u$) and the MDM of the down quark ($\mu_d$) in terms of the proton charge $e$ and the quark masses $m_u$ and $m_d$?

(b) If the neutron were an elementary particle, what would be the naive value of the neutron magnetic dipole moment $\mu_n(\text{naive})$?

(c) The neutron has a measured magnetic dipole moment

$$\mu_{\text{neutron}} = \frac{1}{3}(-\mu_u + 4\mu_d) = \mu_n\mu_N, $$

(2)

where $m_p = 938$ MeV and

$$\mu_N = \frac{e\hbar}{2m_pc}. $$

Find the value of $\mu_n$ with effective quark masses $m_u = m_d = 336$ MeV.
Problem (2)

In the energy basis of a harmonic oscillator, we have

\[ X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \quad \text{and} \]
\[ P = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger) \]

where \(a\) and \(a^\dagger\) are the lowering and raising operators, respectively.

Let us define the operator \(a^\dagger a\) as the number operator

\[ N \equiv a^\dagger a \]

and the Hamiltonian can be expressed as

\[ H = \hbar\omega(N + \frac{1}{2}) . \]

The eigenvector is \(|n\rangle\) such that

\[ N|n\rangle = n|n\rangle \]

where \(n\) is the eigenvalue and the vectors \(|n\rangle\) form a set of orthonormal basis vectors. Then we have

\[ H|n\rangle = \hbar\omega(N + \frac{1}{2})|n\rangle = E_n|n\rangle \]

with

\[ E_n = (n + \frac{1}{2})\hbar\omega . \]

(a) and (b) Find \(c_n\) and \(d_n\) for lowering and raising operators:

(a) \(a|n\rangle = c_n|n - 1\rangle\), and

(b) \(a^\dagger|n\rangle = d_n|n + 1\rangle\).

(c) and (d) Calculate the following matrix elements for a quantum oscillator:

(c) \(\langle 2|X^3|0\rangle\), and

(d) \(\langle 2|X^3|1\rangle\).
Problem (3)

Let us consider three $3 \times 3$ matrices $G_i$ with elements given by

$$(G_i)_{jk} = -i\hbar\epsilon_{ijk}\quad i, j, k = 1, 2, 3$$

where $j$ and $k$ are the row and column indices.

(a) Show that $G_i$’s satisfy the angular momentum commutation relations:

$$[G_i, G_j] = i\hbar\epsilon_{ijk}G_k.$$ 

(b) Find the matrix $G_3$, then calculate the eigenvalues $\lambda_i$ and normalized eigenvectors $|\lambda_i\rangle$ of $G_3$.

(c) Find a unitary matrix that transforms $G_3$ to $J_3$ for $j = 1$ with $J_3$ being diagonal

$$J_3 = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$ 

N.B. This unitary matrix transforms the Cartesian space representation of the angular momentum operator $G_i$ to its spherical basis representation $J_i$ for $j = 1$. This problem is helpful in understanding the spin of photon.

Problem (4)

The Pauli spin matrices are defined as

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

(a) Three two-by-two matrices $(t_i, i = 1, 2, 3)$ satisfy the relations

$$[t_1, t_2] = -it_3, \quad [t_2, t_3] = -it_1, \quad [t_3, t_1] = -it_2.$$ 

Determine such a set of matrices.

(b) Show that

$$e^{i\pi\sigma_z/2} = i\sigma_z.$$ 

(c) Show that

$$U(\theta) = e^{-i\tilde{\theta} \cdot \vec{\sigma}/2} = \cos \frac{\theta}{2} - i(\tilde{\theta} \cdot \vec{\sigma}) \sin \frac{\theta}{2}.$$ 

where $\tilde{\theta} = \theta\hat{\theta}$ and $\hat{\theta}$ is the unit vector.

N.B. For a quantum operator, an expression of this form is to be interpreted as

$$e^\Omega = 1 + \Omega + \frac{1}{2} \Omega^2 + \frac{1}{3!} \Omega^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} \Omega^n.$$