PHYS 5213/4213: Nuclear and Particle Physics, Autumn 2021

Problem Set 7 – due November 03

Problem (1)

The magnitude of the magnetic dipole moment (MDM) of an elementary fermion such as a quark of charge q, mass m, and spin s is

$$\vec{\mu} = \left(\frac{q\hbar}{mc}\right)\vec{s}\,. \tag{1}$$

(a) What are the values of the MDM of the up quark (μ_u) and the MDM of the down quark (μ_d) in terms of the proton charge e and the quark masses m_u and m_d?

(b) If the neutron were an elementary particle, what would be the naive value of the neutron magnetic dipole moment $\mu_n(naive)$?

(c) The neutron has a measured magnetic dipole moment

$$\mu_{neutron} = \frac{1}{3}(-\mu_u + 4\mu_d) = \mu_n\mu_N, \tag{2}$$

where $m_p = 938$ MeV and

$$\mu_N = \frac{e\hbar}{2m_pc}\,.$$

Find the value of μ_n with effective quark masses $m_u = m_d = 336$ MeV.

Problem (2)

In the energy basis of a harmonic oscillator, we have

$$\begin{aligned} X &= \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \quad \text{and} \\ P &= -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger) \end{aligned}$$

where a and $a^\dagger$ are the lowering and raising operators, respectively.

Let us define the operator $a^\dagger a$ as the number operator

$$N \equiv a^\dagger a$$

and the Hamiltonian can be expressed as

$$H = \hbar\omega(N + \frac{1}{2}) .$$

The eigenvector is $|n\rangle$ such that

$$N|n\rangle = n|n\rangle$$

where n is the eigenvalue and the vectors $|n\rangle$ form a set of orthonormal basis vectors. Then we have

$$H|n\rangle = \hbar\omega(N + \frac{1}{2})|n\rangle = E_n|n\rangle$$

with

$$E_n = (n + \frac{1}{2})\hbar\omega .$$

(a) and (b) Find c_n and d_n for lowering and raising operators:

(a) $a|n\rangle = c_n|n-1\rangle$, and

(b) $a^\dagger|n\rangle = d_n|n+1\rangle$.

(c) and (d) Calculate the following matrix elements for a quantum oscillator:

(c) $\langle 2|X^3|0\rangle$, and

(d) $\langle 2|X^3|1\rangle$.

Problem (3)

Let us consider three 3×3 matrices G_i with elements given by

$$(G_i)_{jk} = -i\hbar\epsilon_{ijk} \quad i, j, k = 1, 2, 3$$

where j and k are the row and column indices.

(a) Show that G_i's satisfy the angular momentum commutation relations: $[G_i, G_j] = i\hbar\epsilon_{ijk}G_k$.

(b) Find the matrix G_3, then calculate the eigenvalues λ_i and normalized eigenvectors $|\lambda_i\rangle$ of G_3.

(c) Find a unitary matrix that transforms G_3 to J_3 for $j = 1$ with J_3 being diagonal

$$J_3 = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

N.B. This unitary matrix transforms the Cartesian space representation of the angular momentum operator G_i to its spherical basis representation J_i for $j = 1$. This problem is helpful in understanding the spin of photon.

Problem (4)

The Pauli spin matrices are defined as

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Three two-by-two matrices ($t_i, i = 1, 2, 3$) satisfy the relations

$$[t_1, t_2] = -it_3, \quad [t_2, t_3] = -it_1, \quad [t_3, t_1] = -it_2.$$

Determine such a set of matrices.

(b) Show that

$$e^{i\pi\sigma_z/2} = i\sigma_z\,.$$

(c) Show that

$$U(\theta) = e^{-i\vec{\theta}\cdot\vec{\sigma}/2} = \cos\frac{\theta}{2} - i(\hat{\theta}\cdot\vec{\sigma})\sin\frac{\theta}{2}\,.$$

where $\vec{\theta} = \theta\hat{\theta}$ and $\hat{\theta} =$ the unit vector.

N.B. For a quantum operator, an expression of this form is to be interpreted as

$$e^{\Omega} = 1 + \Omega + \frac{1}{2}\Omega^2 + \frac{1}{3!}\Omega^3 + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}\Omega^n\,.$$