Problem Set 6–Due March 12, 2021

Problem (1)
Find the normalization constant \( N \) for
\[
\psi(x) = N \exp\left[ -\frac{1}{2c\hbar}(x - \langle X \rangle)^2 + \frac{i}{\hbar} \langle P \rangle x \right]
\]
such that
\[
\int dx |\psi(x)|^2 = 1
\]
where \( c \) is a constant.

Problem (2) [Griffiths 1.17]
A particle is represented at time \( t = 0 \) by the following wave function
\[
\psi(x) = \begin{cases} 
  A(a^2 - x^2) & \text{for } -a \leq x \leq a, \\
  0 & \text{for } x^2 > a^2,
\end{cases}
\]
(a) Determine the normalization constant \( A \).
(b) What is the expectation value of \( X \) at time \( t = 0 \)?
(c) What is the expectation value of \( P \) at time \( t = 0 \)?
(d) Find the the expectation value of \( X^2 \).
(e) Find the the expectation value of \( P^2 \).
(f) Find the uncertainty (\( \Delta X \)) in \( X \).

Problem (3) [Griffiths 3.31, Virial theorem]
Apply the Ehrenfest's Theorem for a quantum operator \( \Omega \)
\[
\frac{d}{dt} \langle \Omega \rangle = \langle [\Omega, H] \psi \rangle + i\hbar \frac{d}{dt} \langle \psi \rangle
\]
to show that
\[
\frac{d}{dt} \langle XP \rangle = 2\langle T \rangle - \langle X \frac{dV}{dx} \rangle,
\]
where \( T \) is the kinetic energy and the Hamiltonian \( H = T + V \). In a stationary state, the left hand side is zero so
\[
2\langle T \rangle = \langle X \frac{dV}{dx} \rangle.
\]
(4) Let us consider a particle bound in a delta function potential in one dimension

\[ V(x) = -\alpha \delta(x), \quad \alpha > 0, \]

where \( \alpha \) is a constant.

(a) Find the wave function that satisfies the equation of motion

\[ H \psi(x) = E \psi(x) \]

\[ H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \]

in terms of \( k = (-2mE/\hbar^2)^{1/2} \), where \( E \) is the eigenvalue of \( H \) and \( \psi(x) \) is continuous at \( x = 0 \).

(b) Calculate the average total energy \( \langle E \rangle \), the average potential energy \( \langle V \rangle \) and then the average kinetic energy \( \langle T \rangle \) of this particle with the relation \( \langle T \rangle = \langle E \rangle - \langle V \rangle \).

(c) Calculate \( \langle T \rangle \) for the previous problem directly as

\[ \int dx \psi^*(x)(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2})\psi(x) \]

and compare with the previous result.