

Physics 5393  
**Problem Set 6–Due October 13, 2011**

Problem (1)

Let us consider the following operators on a Hilbert space  $V^3$ :

$$L_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(i) Consider the state

$$|\psi\rangle = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix}$$

in the  $L_z$  basis. If  $L_z^2$  is measured in this state and a result  $+1$  is obtained, what is the state after the measurement? How probable was this result? If  $L_z$  is measured, what are the outcomes and respective probabilities?

(ii) A particle is in a state for which the probabilities are

$$\begin{aligned} P(\ell_z = 1) &= 1/4, \\ P(\ell_z = 0) &= 1/2, \text{ and} \\ P(\ell_z = -1) &= 1/4. \end{aligned}$$

The most general normalized state with this property is

$$|\psi\rangle = \frac{e^{i\delta_1}}{2}|\ell_z = 1\rangle + \frac{e^{i\delta_2}}{\sqrt{2}}|\ell_z = 0\rangle + \frac{e^{i\delta_3}}{2}|\ell_z = -1\rangle$$

If  $|\psi\rangle$  is a normalized state then the state  $e^{i\theta}|\psi\rangle$  is a physically equivalent normalized state. Does this mean that the factors  $e^{i\delta_i}$  are irrelevant? Calculate  $P(\ell_x = 1)$ ,  $P(\ell_x = 0)$ , and  $P(\ell_x = -1)$ , for

- (a)  $\delta_1 = \delta_2 = \delta_3 = 0$ ,
- (b)  $\delta_1 = \delta_3 = 0$ , and  $\delta_2 = \pi$ ,
- (c)  $|\psi'\rangle = e^{-i\delta_1}|\psi\rangle$ .

Problem (2)

For a **free particle in one dimension**, the Schrödinger Equation is

$$i\hbar \frac{d}{dt} |\Psi\rangle = H |\Psi\rangle \quad \text{with} \quad H = \frac{P^2}{2m}.$$

(a) Show that the wave function  $\phi(p)$  in the momentum space or the  $p$ -basis is

$$\phi(p) = \langle p | \Psi \rangle = N e^{-i/\hbar \left( \frac{p^2}{2m} \right) t}$$

where  $N$  is the normalization constant.

(b) Find the wave function  $\Psi(x, t) = \langle x | \Psi \rangle$  in the coordinate space or the  $x$ -basis by applying the Fourier transform

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \phi(\hbar k) dk$$

where  $\phi(p) = \phi(\hbar k)$  with  $k = p/\hbar$ .

Problem (3)

For a particle moving in an **infinite square well** of width  $2a$  with the potential energy

$$V(x) = \begin{cases} 0, & \text{for } -a \leq x \leq a \text{ with } a > 0, \text{ and} \\ \infty, & \text{otherwise,} \end{cases}$$

its normalized wave function inside the well at time  $t = 0$  is

$$\Psi(x, 0) = C \left[ \sin \frac{\pi x}{a} + \frac{1}{4} \cos \frac{3\pi x}{2a} \right]$$

and  $\Psi(x, 0) = 0$  for  $x^2 \geq a^2$ .

- (a) Calculate the coefficient  $C$ .
- (b) What is the wave function  $\Psi(x, t)$ ?
- (c) If a measurement of total energy is made, what are the possible results of such a measurement, and what is the probability to measure each of them?
- (d) What is the expectation value of the energy  $\langle E \rangle$ ?

Problem (4)

Let us consider a one-dimensional quantum harmonic oscillator with the Hamiltonian

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2, \quad \text{and}$$
$$H|n\rangle = E_n|n\rangle.$$

- (a) Find  $\langle X \rangle$ ,  $\langle P \rangle$ ,  $\langle X^2 \rangle$ ,  $\langle P^2 \rangle$  and  $\Delta X \Delta P$  in the state  $|n\rangle$ .
- (b) What is the uncertainty relation for the ground state.