PHYSICS 6433

Problem Set 6 – Due March 22, 2017

Problem (1): Real Scalar Fields

In the energy basis, the Hamiltonian of a real scalar field is

$$H = \int \frac{d^3k}{(2\pi)^3} \frac{\omega_k}{2} \left[a(\vec{k}) a^\dagger(\vec{k}) + a^\dagger(\vec{k}) a(\vec{k}) \right] .$$

Show that

$$\text{(a)} \ [a(\vec{k}), H] = \omega_k a(\vec{k})$$

and

$$\text{(b)} \ [a^\dagger(\vec{k}), H] = -\omega_k a^\dagger(\vec{k}) .$$

This shows that the operators $a(\vec{k})$ and $a^\dagger(\vec{k})$ annihilate and create a particle of energy ω_k respectively.

Problem (2): The Bridge between Coordinate Space and Momentum Space

(a) **Quantum Mechanics**

In quantum mechanics, the eigenvalue equation for the momentum operator is

$$K|k\rangle = k|k\rangle .$$

In the x-basis, this equation becomes

$$\langle x|K|k\rangle = k\langle x|k\rangle$$

where the matrix element of K in the x-basis is

$$K_{xy} = \langle x|K|y\rangle = -i\frac{d}{dx}\delta(x-y) .$$

Show that the bridge between the x-basis and the k-basis is

$$\langle x|k\rangle = \psi_k(x) = Ae^{ikx}$$

and find the normalization constant A for $\psi_k(x)$ with $\hbar = 1$ and

$$\langle p|q\rangle = (2\pi)\delta(p-q)$$

in one dimension.

(b) **Quantum Field Theory with a Real Scalar Field**

For a quantized real scalar field $\phi(x)$, the state

$$|\phi(x)\rangle \equiv \phi(x)|0\rangle = \phi^{(-)}(x)|0\rangle = \int \frac{d^3q}{(2\pi)^3}\frac{1}{2E_q}e^{iqx}|q\rangle$$

can be thought of as a state for a single particle at the coordinate x. Except for the factor $1/(2E_p)$, this state $|\phi(x)\rangle$ is the same as the eigenstate of position $|x\rangle$.

Show that

$$\langle\phi(x)|p\rangle = \langle 0|\phi(x)|p\rangle = e^{-ipx} = e^{-ip^0x^0+i\vec{p}\cdot\vec{x}}\,.$$

We can interpret this as the position space representation of the single particle wavefunction of the state $|p\rangle$.

Problem (3): Lorentz Transformations

(a) Show by an explicit computation that Lorentz boosts do not commute.

(b) Show that Lorentz boosts and rotations do not commute.

(c) Which subgroups of the Lorentz group do in fact commute, namely, what are the Abelian subgroups?

Problem (4): Lorentz Algebra

Show that the matrices

$$(T^{\mu\nu})_{\alpha\beta} = i(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha)$$

and

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu,\gamma^\nu]$$

satisfy the Lorentz algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho} + g^{\mu\sigma}M^{\nu\rho})\,.$$