

Physics 5403

Problem Set 5 – Due October 2, 2009

(1). Here are some useful tricks for evaluating the expectation values of certain operators in the eigenstates of hydrogen atom.

- (a) Suppose we want $\langle 1/R \rangle_{n\ell m}$. Consider first $\langle \lambda/R \rangle$. We can interpret $\langle \lambda/R \rangle$ as the first order correction due to a perturbation λ/R . Now this problem can be solved exactly; we just replace e^2 with $e^2 - \lambda$ everywhere. So the exact energy is

$$\begin{aligned} E_n(\lambda) &= -\frac{m(e^2 - \lambda)^2}{2\hbar^2 n^2} \\ &= E_n^{(0)} + E_n^{(1)} + E_n^{(2)} + \dots \\ &= E_n(\lambda = 0) + \lambda \frac{dE_n}{d\lambda} \Big|_{\lambda=0} + O(\lambda^2) \end{aligned}$$

Find the first order correction ($E_n^{(1)}$) that is the term linear in λ and then find $\langle 1/R \rangle$.

- (b) Consider now $\langle \lambda/R^2 \rangle$. In this case, an exact solution is possible since the perturbation just modifies the centrifugal term as follows:

$$\frac{\hbar^2 \ell(\ell + 1)}{2mr^2} + \frac{\lambda}{r^2} = \frac{\hbar^2 \ell'(\ell' + 1)}{2mr^2}$$

where ℓ' is a function of λ . Now the dependence of E_n on $\ell'(\lambda)$ is

$$\begin{aligned} E_n(\lambda) &= -\frac{me^4}{2\hbar^2(k + \ell' + 1)^2} \\ &= E_n^{(0)} + E_n^{(1)} + E_n^{(2)} + \dots \\ &= E_n(0) + \lambda \frac{dE_n}{d\lambda} \Big|_{\lambda=0} + O(\lambda^2) \end{aligned}$$

Find the first order correction ($E_n^{(1)}$) that is the term linear in λ and then find $\langle 1/R^2 \rangle$.

- (c) Consider finally $\langle 1/R^3 \rangle$. Since there is no such term in the Coulomb Hamiltonian, we consider the radial momentum operator,

$$P_r = -i\hbar \left(\frac{\partial}{\partial r} + \frac{1}{r} \right)$$

and the radial part of the Hamiltonian becomes

$$\frac{P_r^2}{2m} = -\frac{\hbar^2}{2m} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) \right]$$

Apply the fact that $\langle [P_r, H] \rangle = 0$ in the energy eigenstates, and by explicitly evaluating the commutator, show that

$$\begin{aligned} \left\langle \frac{1}{R^3} \right\rangle_{n\ell m} &= \frac{1}{a_0[\ell(\ell+1)]} \left\langle \frac{1}{R^2} \right\rangle_{n\ell m} \\ &= \frac{1}{a_0^3 n^3 [\ell(\ell+1/2)(\ell+1)]} \end{aligned}$$

where

$$a_0 = \frac{\hbar^2}{me^2}$$

is the Bohr radius of the Hydrogen atom.

(2). The Hamiltonian matrix for a two-state system can be written as

$$H = \begin{pmatrix} E_1^0 & \lambda\Delta \\ \lambda\Delta & E_2^0 \end{pmatrix}$$

The energy eigenvectors for the unperturbed Hamiltonian ($\lambda = 0$) are given by

$$|\psi_1^{(0)}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\psi_2^{(0)}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

- (a) Solve this problem exactly to find the energy eigenvalues E_1 and E_2 as well as the energy eigenvectors $|\psi_1\rangle$ and $|\psi_2\rangle$.
- (b) Assuming that $\lambda|\Delta| \ll |E_1^0 - E_2^0|$, solve the same problem by using time-independent perturbation theory up to first order in the energy eigenvectors and up to second order in the energy eigenvalues. Compare them with the exact results obtained in part (a).
- (c) Suppose the two unperturbed energies are “almost degenerate,” that is

$$|E_1^0 - E_2^0| \ll \lambda|\Delta|.$$

Show that the exact results obtained in (a) closely resemble what you would expect by applying degenerate perturbation theory to this problem with E_1^0 set exactly equal to E_2^0 .

(3). Let us consider a one-dimensional simple harmonic oscillator whose classical angular frequency is ω_0 . For $t < 0$, it is known to be in the ground state. For $t > 0$, there is an additional time-dependent potential

$$H_1(t) = V(t) = F_0 \cos(\omega t) X$$

where F_0 is constant in both space and time. Find an expression for the expectation value $\langle X \rangle$ as a function of time by using time-dependent perturbation theory to lowest nonvanishing order. Is this procedure valid for $\omega \simeq \omega_0$?

Hint: You may use $\langle k|X|n\rangle = \sqrt{\hbar/2m\omega_0} (\sqrt{n+1} \delta_{k,n+1} + \sqrt{n} \delta_{k,n-1})$.