

Physics 5393  
**Problem Set 5—Due October 6, 2011**

Problem (1): Normalization of Wave Functions

- (a) Find the normalization constant  $A$  for

$$\psi(x) = A \exp\left[-\frac{1}{2c\hbar}(x - \langle X \rangle)^2 + \frac{i}{\hbar}\langle P \rangle x\right]$$

such that

$$\int |\psi(x)|^2 dx = 1$$

where  $c$  is a constant.

- (b) Calculate the Fourier transform  $g(\vec{k})$  of the function

$$f(\vec{x}) = N e^{-r^2/a^2} e^{-\mu r}, \quad r = |\vec{x}|$$

in the 3-dimensional space. Find the normalization constant  $N$  for  $f(\vec{r})$  and an exact expression of  $g(\vec{k})$  for  $\mu = 0$ . For  $\mu \neq 0$ , express  $g(\vec{k})$  as an integral.

Problem (2): Expectation Value and Uncertainty

A particle is represented at time  $t = 0$  by the following wave function

$$\psi(x) = \begin{cases} A(a^2 - x^2) & \text{for } -a \leq x \leq a, \quad a > 0, \text{ and} \\ 0, & \text{for } x^2 > a^2, \end{cases}$$

- (a) Determine the normalization constant  $A$ .
- (b) What is the expectation value of  $X$  at time  $t = 0$ ?
- (c) What is the expectation value of  $P$  at time  $t = 0$ ?
- (d) Find the the expectation value of  $X^2$ .
- (e) Find the the expectation value of  $P^2$ .
- (f) Find the uncertainty  $(\Delta X)$  in  $X$ .

Problem (3): Ehrenfest's Theorem

- (a) Find  $d\langle P \rangle/dt$  by applying the Ehrenfest's Theorem. Then compare your result with Hamilton's equation for  $dp/dt$ .
- (b) Apply the Ehrenfest's Theorem for a quantum operator  $\Omega$

$$i\hbar \frac{d}{dt} \langle \Omega \rangle = \langle \psi | [\Omega, H] | \psi \rangle + i\hbar \langle \frac{\partial}{\partial t} \Omega \rangle$$

to show that

$$\frac{d}{dt} \langle XP \rangle = 2\langle T \rangle - \langle X \frac{dV}{dx} \rangle,$$

where  $T$  is the kinetic energy and the Hamiltonian  $H = T + V$ . In a stationary state, the left hand side is zero so

$$2\langle T \rangle = \langle X \frac{dV}{dx} \rangle.$$

Problem (4): Schrödinger Equation in One Dimension

Let us consider a particle bound in a delta function potential in one dimension

$$V(x) = -\alpha\delta(x), \quad \alpha > 0,$$

where  $\alpha$  is a constant.

- (a) Find the wave function that satisfies the equation of motion

$$\begin{aligned} H\psi(x) &= E\psi(x) \\ H &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \end{aligned}$$

in terms of  $k = (-2mE/\hbar^2)^{1/2}$ , where  $E$  is the eigenvalue of  $H$  and  $\psi(x)$  is continuous at  $x = 0$ .

- (b) Calculate the average total energy  $\langle E \rangle$ , the average potential energy  $\langle V \rangle$  and then the average kinetic energy  $\langle T \rangle$  of this particle with the relation  $\langle T \rangle = \langle E \rangle - \langle V \rangle$ .
- (c) Calculate  $\langle T \rangle$  for the previous problem directly as

$$\int dx \psi^*(x) \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \right) \psi(x)$$

and compare with the previous result.