PHYS 5213/4213: Nuclear and Particle Physics, Autumn 2021

Problem Set 5 – due October 20

Problem (1)

A muon at rest decays into an electron and two neutrinos $\mu^- \to e^- \bar{\nu}_e \nu_\mu$. Find the maximum energy and the maximum momentum for the electron (E_e^{max} and $|\vec{p}_e|^{max}$) in terms of m_μ and m_e with $m_\nu = 0$.

Problem (2)

For a system with a spherically symmetric potential, the complete solution to the Schrödinger equation is

$$\psi_{\ell,m}(r,\theta,\phi) = R(r) Y_{\ell,m}(\theta,\phi)$$

where

$$Y_{\ell,m} = \epsilon \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}} P_{\ell,m}(\cos\theta) e^{im\phi}$$

with $\epsilon = (-1)^m$ for $m > 0$ and $\epsilon = +1$ for $m < 0$,

$$P_{\ell,m}(z) = (1-z^2)^{\frac{|m|}{2}} \frac{d^{|m|}}{dz^{|m|}} P_\ell(z)$$

with $z = \cos\theta$ and

$$P_\ell(z) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dz^\ell} (z^2-1)^\ell.$$

Parity means reflecting a vector through the origin. In spherical coordinates, a position vector is described with $\vec{r} = (r,\theta,\phi)$. Find (a) $\Pi e^{im\phi}$, (b) $\Pi P_{\ell,m}(z)$, and (c) $\Pi Y_{\ell,m}(\theta,\phi)$, where $\Pi =$ the parity operator.

Problem (3)

An element of a $SU(N)$ group is often expressed as

$$U(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{G}} = e^{i\alpha_i G_i},$$

and it is represented with a $N \times N$ matrix.

Show that each generator G_i, (a) is traceless, (b) is Hermitian, and (c) it has $N^2 - 1$ independent parameters. *Hint:* You might need to use

$$\det U(\vec{\alpha}) = e^{\mathrm{Tr}\ln U(\vec{\alpha})} = e^{i\mathrm{Tr}(\vec{\alpha}\cdot\vec{G})}.$$

Problem (4)

In the fundamental representation, a general vector is a doublet

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix},$$

and it becomes ψ' under a unitary SU(2) transformation

$$\psi' = U(\alpha)\psi\,, \quad \text{with} \quad U(\alpha) = e^{i\alpha_i t_i}\,, \quad i = 1, 2, 3,$$

where $t_i = (1/2)\sigma_i$.

The complex conjugate representation has state vectors

$$\psi^* = \begin{pmatrix} (\psi^1)^* \\ (\psi^2)^* \end{pmatrix},$$

and they transform as

$$\psi^* \to (\psi^*)' = (\psi')^* = U^*\psi^*\,, \quad \text{where} \quad \psi^* = \begin{pmatrix} (\psi^1)^* \\ (\psi^2)^* \end{pmatrix} \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

Let us study the special property of the SU(2) complex conjugate representation with an infinitesimal transformation

$$U(\epsilon) = I + i\epsilon_i t_i\,, \quad \epsilon_i \to 0+\,.$$

(a) For every 2×2 unitary matrix U with unit determinant, show that there exists a matrix S which connects U to its complex conjugate matrix U^* through the similarity transformation

$$S^{-1}US = U^*\,,$$

such that

$$\tilde{\psi}' = U(\alpha)\tilde{\psi} \quad \text{where} \quad \tilde{\psi} \equiv S\psi^*\,.$$

(b) Suppose $\psi^i, i = 1, 2$ are the bases for the fundamental representation of SU(2) with eigenvalue equations

$$t_3\psi^1 = \frac{1}{2}\psi^1 \quad \text{and} \quad t_3\psi^2 = -\frac{1}{2}\psi^2\,.$$

Evaluate the eigenvalues of t_3 operating on $(\psi^1)^*$ and $(\psi^2)^*$ respectively.