PHYSICS 6433

Problem Set 4 – Due February 16, 2017

Problem (1): Noether's Theorem in Classical Mechanics

For every continuous symmetry of the form $q_i \to z_i(q, \alpha)$, such that $L(z, \dot{z}) = L(q, \dot{q})$, there is a conservation law for the conserved quantity

$$Q = \sum_i \Lambda_i \frac{\partial L}{\partial \dot{q}_i} = \text{ constant},$$

where $\alpha =$ constant,

$$\Lambda_i \equiv \frac{\partial z_i(q, \alpha)}{\partial \alpha}|_{\alpha=0}, \text{ and } \dot{\Lambda}_i = \frac{\partial \dot{z}_i}{\partial \alpha}|_{\alpha=0}.$$

(a). For a rotation $\theta \to \phi = \theta + \alpha$ with a constant α, find Λ and $\dot{\Lambda}$.

(b). Find the conserved quantity Q associated with rotational invariance under a transformation $\phi = \theta + \alpha$ for the Lagrangian $L(r, \dot{r}, \theta, \dot{\theta})$ with a central force.

Problem (2): Noether's Theorem in Field Theory

Let us consider a continuous transformation of the form

$$\begin{aligned} x^\mu &\to x'^\mu, \ \ \phi(x) \to \phi'(x'), \ \ \partial_\mu \phi(x) \to \partial'_\mu \phi'(x') \\ \delta\phi(x) &\equiv \phi'(x) - \phi(x) \\ \delta(\partial_\mu \phi(x)) &= \partial_\mu(\delta\phi(x)) = \partial_\mu \phi'(x) - \partial_\mu \phi(x) \end{aligned}$$

where $\phi(x)$ is a scalar field such that $\phi'(x') = \phi(x)$.

If the action (S) is invariant under a continuous transformation, then we have

$$\begin{aligned} \delta S &= \int d^4x' \mathcal{L}\left(\phi'(x'), \partial'_\mu \phi'(x')\right) - \int d^4x \mathcal{L}\left(\phi(x), \partial_\mu \phi(x)\right) \\ &= \int d^4x \left[\mathcal{L}\left(\phi'(x), \partial_\mu \phi'(x)\right) - \mathcal{L}\left(\phi(x), \partial_\mu \phi(x)\right)\right] \\ &\quad + \int W d^4x \left[\mathcal{L}\left(\phi(x), \partial_\mu \phi(x)\right) + \frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu\right] - \int d^4x \mathcal{L}\left(\phi(x), \partial_\mu \phi(x)\right) \end{aligned}$$

where W is the Jacobian of the coordinate transformation.

Let us consider an infinitesimal transformation with

$$W = \left|\frac{\partial x'}{\partial x}\right| = 1 + \partial_\mu(\delta x^\mu)$$

then we have

$$\delta S = \int d^4x \left[\mathcal{L}\left(\phi'(x), \partial_\mu \phi'(x)\right) - \mathcal{L}\left(\phi(x), \partial_\mu \phi(x)\right)\right] - \int d^4x (\partial_\mu K^\mu) = 0 \ .$$

(a) Find K^μ, then apply the Euler-Lagrange equation to show that there exists a conserved current

$$J^\mu(x) = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\right) \delta\phi(x) - K^\mu$$

such that $\partial_\mu J^\mu = 0$.

(b) Let us consider an invariant action of a scalar field $\phi(x)$ under an infinitesimal space-time translation

$$x'^\mu = x^\mu + \epsilon^\mu \ \text{ and } \ \delta\phi(x) = -\epsilon^\mu \partial_\mu \phi(x) = -\epsilon_\nu \partial^\nu \phi(x)$$

or $\delta x^\mu = \epsilon^\mu =$ infinitesimal constant. Show that

$$K^\mu = -\epsilon^\mu \mathcal{L}$$

and

$$J^\mu = -\epsilon_\nu T^{\mu\nu}$$

where

$$T^{\mu\nu} \equiv \partial^\nu \phi(x) \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \right) - g^{\mu\nu} \mathcal{L} \, .$$

This is known as the energy-momentum stress tensor of the theory.

Problem (3): The Hamiltonian of a Scalar Field

In the case of free Klein-Gordon theory with quantized fields, the Hamiltonian is

$$\begin{aligned} H &= \int d^3x \mathcal{H} \\ &= \int d^3x \left(\frac{1}{2}\pi^2(x) + \frac{1}{2}\nabla\phi \cdot \nabla\phi + \frac{1}{2}m^2\phi^2 \right) \end{aligned}$$

where

$$\begin{aligned} \phi(x) &= \int \frac{d^3k}{(2\pi)^3\sqrt{2k^0}} \left[a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x} \right] \quad \text{and} \\ \pi(x) &= -i \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{k^0}{2}} \left[a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x} \right] \, . \end{aligned}$$

The operators $a(\vec{k})$ and $a^\dagger(\vec{k})$ have commutation relations analogous to the annihilation and creation operators of a harmonic oscillator:

$$[a(\vec{k}), a(\vec{q})] = 0 = [a^\dagger(\vec{k}), a^\dagger(\vec{q})] \;\text{ and }\; [a(\vec{k}), a^\dagger(\vec{q})] = (2\pi)^3\delta^3(k - q) \, .$$

(a) Show that

$$\begin{aligned} \int d^3x \pi^2(x) &= -\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} (k^0)[-a(\vec{k})a^\dagger(\vec{k}) - a^\dagger(\vec{k})a(\vec{k}) \\ &\quad + e^{-2ik^0x^0} a(\vec{k})a(-\vec{k}) + e^{2ik^0x^0} a^\dagger(\vec{k})a^\dagger(-\vec{k})] \, . \end{aligned}$$

(b) Show that

$$\begin{aligned} \int d^3x \nabla\phi(x) \cdot \nabla\phi(x) &= -\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} (\frac{|\vec{k}|^2}{k^0})[-a(\vec{k})a^\dagger(\vec{k}) - a^\dagger(\vec{k})a(\vec{k}) \\ &\quad - e^{-2ik^0x^0} a(\vec{k})a(-\vec{k}) - e^{2ik^0x^0} a^\dagger(\vec{k})a^\dagger(-\vec{k})] \, . \end{aligned}$$

(c) Show that

$$\begin{aligned} \int d^3x \phi^2(x) &= \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^0} [a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k}) \\ &\quad + e^{-2ik^0x^0} a(\vec{k})a(-\vec{k}) + e^{2ik^0x^0} a^\dagger(\vec{k})a^\dagger(-\vec{k})]; . \end{aligned}$$

(d) Substitute (a)-(c) into the Hamiltonian, and show that

$$H = \int \frac{d^3k}{(2\pi)^3} \frac{\omega_k}{2} \left[a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k}) \right] \, .$$