Problem (1): Noether’s Theorem in Classical Mechanics

For every continuous symmetry of the form \( q_i \rightarrow z_i(q, \alpha) \), such that \( L(z, \dot{z}) = L(q, \dot{q}) \), there is a conservation law for the conserved quantity

\[
Q = \sum_i \Lambda_i \frac{\partial L}{\partial \dot{q}_i} = \text{constant},
\]

where \( \alpha = \text{constant} \),

\[
\Lambda_i \equiv \left. \frac{\partial z_i(q, \alpha)}{\partial \alpha} \right|_{\alpha=0}, \quad \text{and} \quad \dot{\Lambda}_i \equiv \left. \frac{\partial \dot{z}_i}{\partial \alpha} \right|_{\alpha=0}.
\]

(a). For a rotation \( \theta \rightarrow \phi = \theta + \alpha \) with a constant \( \alpha \), find \( \Lambda \) and \( \dot{\Lambda} \).

(b). Find the conserved quantity \( Q \) associated with rotational invariance under a transformation \( \phi = \theta + \alpha \) for the Lagrangian \( L(r, \dot{r}, \theta, \dot{\theta}) \) with a central force.
Problem (2): Noether’s Theorem in Field Theory

Let us consider a continuous transformation of the form

\[
x^\mu \to x'^\mu, \quad \phi(x) \to \phi'(x') \quad \partial_\mu \phi(x) \to \partial'_\mu \phi'(x')
\]

\[
\delta \phi(x) \equiv \phi'(x) - \phi(x)
\]

\[
\delta (\partial_\mu \phi(x)) = \partial'_\mu (\delta \phi(x)) = \partial'_\mu \phi'(x) - \partial_\mu \phi(x)
\]

where \( \phi(x) \) is a scalar field such that \( \phi'(x') = \phi(x) \).

If the action \( S \) is invariant under a continuous transformation, then we have

\[
\delta S = \int d^4x' \mathcal{L} \left( \phi'(x'), \partial_\mu' \phi'(x') \right) - \int d^4x \mathcal{L} \left( \phi(x), \partial_\mu \phi(x) \right)
\]

\[
= \int d^4x \left[ \mathcal{L} \left( \phi'(x), \partial_\mu' \phi(x) \right) - \mathcal{L} \left( \phi(x), \partial_\mu \phi(x) \right) \right]
\]

\[
+ \int W d^4x \left[ \mathcal{L} \left( \phi(x), \partial_\mu \phi(x) \right) + \frac{\partial \mathcal{L}}{\partial \partial_\mu x} \delta x^\mu \right] - \int d^4x \mathcal{L} \left( \phi(x), \partial_\mu \phi(x) \right)
\]

where \( W \) is the Jacobian of the coordinate transformation.

Let us consider an infinitesimal transformation with

\[
W = \left| \frac{\partial x'}{\partial x} \right| = 1 + \partial_\mu (\delta x^\mu)
\]

then we have

\[
\delta S = \int d^4x \left[ \mathcal{L} \left( \phi'(x), \partial_\mu' \phi(x) \right) - \mathcal{L} \left( \phi(x), \partial_\mu \phi(x) \right) \right] - \int d^4x (\partial_\mu K^\mu) = 0.
\]

(a) Find \( K^\mu \), then apply the Euler-Lagrange equation to show that there exists a conserved current

\[
J^\mu(x) = \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \right) \delta \phi(x) - K^\mu
\]

such that \( \partial_\mu J^\mu = 0 \).

(b) Let us consider an invariant action of a scalar field \( \phi(x) \) under an infinitesimal space-time translation

\[
x'^\mu = x^\mu + \epsilon^\mu \quad \text{and} \quad \delta \phi(x) = - \epsilon^\mu \partial_\mu \phi(x) = - \epsilon_\nu \partial'_\nu \phi(x)
\]

or \( \delta x^\mu = \epsilon^\mu = \text{infinitesimal constant} \). Show that

\[
K^\mu = - \epsilon^\mu \mathcal{L}
\]

and

\[
J^\mu = - \epsilon_\nu T^{\mu\nu}
\]
where
\[ T^{\mu\nu} \equiv \partial^\mu \phi(x) \left( \frac{\partial L}{\partial (\partial_\mu \phi(x))} \right) - g^{\mu\nu} L. \]

This is known as the energy-momentum stress tensor of the theory.

**Problem (3): The Hamiltonian of a Scalar Field**

In the case of free Klein-Gordon theory with quantized fields, the Hamiltonian is
\[ H = \int d^3x \mathcal{H} = \int d^3x \left( \frac{1}{2} \pi^2(x) + \frac{1}{2} \nabla \phi \cdot \nabla \phi + \frac{1}{2} m^2 \phi^2 \right) \]
where
\[ \phi(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2k^0}} \left[ a(\vec{k}) e^{-ik \cdot x} + a^\dagger(\vec{k}) e^{ik \cdot x} \right] \quad \text{and} \quad \pi(x) = -i \int \frac{d^3k}{(2\pi)^3 \sqrt{2k^0}} \left[ a(\vec{k}) e^{-ik \cdot x} - a^\dagger(\vec{k}) e^{ik \cdot x} \right]. \]

The operators \( a(\vec{k}) \) and \( a^\dagger(\vec{k}) \) have commutation relations analogous to the annihilation and creation operators of a harmonic oscillator:
\[ [a(\vec{k}), a(\vec{q})] = 0 = [a^\dagger(\vec{k}), a^\dagger(\vec{q})] = (2\pi)^3 \delta^3(k - q). \]

(a) Show that
\[ \int d^3x \pi^2(x) = -\frac{1}{2} \int \frac{d^3k}{(2\pi)^3 (k^0)} \left[ -a(\vec{k}) a^\dagger(\vec{k}) - a^\dagger(\vec{k}) a(\vec{k}) + e^{-2ik^0 x^0} a(\vec{k}) a(-\vec{k}) + e^{2ik^0 x^0} a^\dagger(\vec{k}) a^\dagger(-\vec{k}) \right]. \]

(b) Show that
\[ \int d^3x \nabla \phi(x) \cdot \nabla \phi(x) = -\frac{1}{2} \int \frac{d^3k}{(2\pi)^3 (k^0)} \left[ -a(\vec{k}) a^\dagger(\vec{k}) - a^\dagger(\vec{k}) a(\vec{k}) - e^{-2ik^0 x^0} a(\vec{k}) a(-\vec{k}) - e^{2ik^0 x^0} a^\dagger(\vec{k}) a^\dagger(-\vec{k}) \right]. \]

(c) Show that
\[ \int d^3x \phi^2(x) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{k^0} a(\vec{k}) a^\dagger(\vec{k}) + a^\dagger(\vec{k}) a(\vec{k}) + e^{-2ik^0 x^0} a(\vec{k}) a(-\vec{k}) + e^{2ik^0 x^0} a^\dagger(\vec{k}) a^\dagger(-\vec{k}) \right]. \]

(d) Substitute (a)-(c) into the Hamiltonian, and show that
\[ H = \int \frac{d^3k}{(2\pi)^3} \frac{\omega_{k}}{2} \left[ a(\vec{k}) a^\dagger(\vec{k}) + a^\dagger(\vec{k}) a(\vec{k}) \right]. \]