PHYS 6433: Quantum Field Theory

Problem Set 3 – Due February 08, 2017

Problem (1): Finite Rotation for a Dirac Spinor

Let us consider a finite rotation of angle ϕ about the $z-$axis with $x' = \Lambda x$ and

$$\omega_{12} = \epsilon_{123}\phi_3 = \phi_3 = \phi \quad \text{where} \quad \omega_{ij} = \epsilon_{ijk}\phi_k$$

Then the Dirac spinor will change under this rotation as $\psi'(x') = S_R(\Lambda)\psi(x)$.

(a) Find the rotation operator Λ as a 4×4 matrix.

(b) Find σ_{ij} for an infinitesimal rotation

$$S_R(\Lambda) = I - \frac{i}{4}\sigma_{ij}\omega^{ij}\,.$$

(c) Show that for a finite rotation of angle ϕ about the $z-$axis,

$$S_R(\Lambda) = e^{-\frac{i}{2}\Sigma_3\phi_3} = I\cos\left(\frac{\phi}{2}\right) - i\Sigma_3\sin\left(\frac{\phi}{2}\right) \quad \text{where} \quad \Sigma_3 = \sigma_{12} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}.$$

(d) Find the rotation operator $S_R(\Lambda)$ as a 4×4 matrix for a Dirac spinor.

Problem (2): Maxwell equations

In classical electrodynamics with the Heaviside-Lorentz conventions and $c = 1$, the 4-vector potential is defined as

$$A^\mu = (\Phi, \vec{A});$$

and the 4-vector current density is defined to be

$$J^\mu \equiv (\rho, \vec{J})\,.$$

Furthermore, let us defined a second-rank antisymmetric field-strength tensor as

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu,$$

then we can express the first two Maxwell equations with sources as

$$\partial_\mu F^{\mu\nu} = \partial_\mu\partial^\mu A^\nu = J^\nu,$$

with the Lorenz condition $\partial_\mu A^\mu = 0$.

Show that

$$\Box A^\nu = J^\nu,$$

that is

$$\begin{aligned} \partial_\mu F^{\mu 0} &= \partial_\mu \partial^\mu A^0 = J^0 = \rho, \quad \text{and} \\ \partial_\mu F^{\mu i} &= \partial_\mu \partial^\mu A^i = J^i, \quad i = 1, 2, 3. \end{aligned}$$

Problem (3): Lagrangian for Vector Particles

The Lagrangian density of electrodynamics is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu$$

for the photon field A^μ. Find the equation of motion for A^μ.

Problem (4): Polarization Vectors for a Massive Vector Particle

For a vector particle of mass M, energy E, and momentum $\vec{k}$ along the z axis, the polarization vectors can be expressed as

$$\begin{aligned} \epsilon^\mu(\lambda = \pm 1) &= \mp \frac{1}{\sqrt{2}}(0, 1, \pm i, 0), \quad \text{and} \\ \epsilon^\mu(\lambda = 0) &= (|\vec{k}|, 0, 0, E)/M \end{aligned}$$

that are called circular polarization vectors.

- Determine how $\vec{\epsilon}(\lambda = 1)$ and $\vec{\epsilon}(\lambda = -1)$ transform under a rotation ϕ about the z axis.

- Show that every polarization vector satisfies the following relations

$$\begin{aligned} \epsilon_\mu k^\mu &= 0 \quad \text{and} \\ \epsilon_\mu^* \epsilon^\mu &= -1 \; . \end{aligned}$$

- Show that the completeness relation is

$$\sum_\lambda \epsilon_\mu^*(\lambda) \epsilon_\nu(\lambda) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2} \; .$$

N.B. There is a minus sign in the definition of $\epsilon(\lambda = +1)$. This is a standard phase convention used in the construction of the spherical harmonics $Y_{\ell m}$ for $\ell = 1$ and $m = \pm 1$.