

Physics 5393

Problem Set 2 – Due September 8, 2011

(1). An operator Ω is linear if

$$\Omega(a\psi_1(x) + b\psi_2(x)) = a(\Omega\psi_1(x)) + b(\Omega\psi_2(x)),$$

where $\psi_1(x)$ and $\psi_2(x)$ are functions of x while a and b are complex numbers. Determine which of the following operators are linear.

(a) $\Omega\phi(x) = \phi(-x),$

(b) $\Omega\phi(x) = \phi(x + c),$

(c) $\Omega\phi(x) = \phi(x) + c,$

(d) $\Omega\phi(x) = \int_{-\infty}^{\infty} K(x, y)\phi(y)dy,$

where $c = \text{constant}$ and $K(x, y) = K^*(y, x)$.

Hint: Consider $\phi(x) = a\psi_1(x) + b\psi_2(x)$, then compare $\Omega\phi(x)$ with $a(\Omega\psi_1(x)) + b(\Omega\psi_2(x))$.

(2). In the Hilbert space, an operator is Hermitian if $\Omega^\dagger = \Omega$ or $\langle \psi | \Omega^\dagger | \phi \rangle = \langle \psi | \Omega | \phi \rangle$.

(a) Show that in the x -basis, an operator is Hermitian if

$$\int [\Omega\psi(x)]^* \phi(x) dx = \int \psi^*(x) [\Omega\phi(x)] dx$$

where $\phi(x) \equiv \langle x | \phi \rangle$ and $\Omega\phi(x) \equiv \langle x | \Omega | \phi \rangle$.

Which of the following operators are Hermitian?

(b) $\Omega\phi(x) = \phi(x + a),$

(c) $\Omega\phi(x) = \phi^*(x),$

(d) $\Omega\phi(x) = \phi(-x),$

(e) $\Omega\phi(x) = \int_{-\infty}^{\infty} K(x, y)\phi(y)dy,$

where $a = \text{constant}$, $K(x, y)$ is real, and $K(x, y) = -K(y, x)$.

(3). Show how to go from the basis

$$|v_1\rangle = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \quad |v_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad |v_3\rangle = \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}$$

to the orthonormal basis

$$|u_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |u_2\rangle = \begin{pmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \quad |u_3\rangle = \begin{pmatrix} 0 \\ -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}.$$

Hint: Apply the Gram-Schmidt Theorem.

(4). Suppose a 2×2 matrix (Ω) is written as

$$\Omega = a_0 \cdot I + \vec{a} \cdot \vec{\sigma} = a_0 + \vec{a} \cdot \vec{\sigma}$$

where I is the identity matrix, σ_k are Pauli matrices, a_0 and $a_k, k = 1, 2, 3$ are numbers.

(a) Evaluate $\text{Tr}(\Omega)$ and $\text{Tr}(\sigma_k \Omega)$.

(b) Obtain a_0 and a_k in terms of matrix elements Ω_{ij} .