Problem 1: Photon propagator in the $R_\xi$ gauge

In QED, the free field generating functional in the covariant gauge is given as

$$Z_0[J] = \int \mathcal{D}A_\mu e^{iS[A_\mu]} = \int \mathcal{D}A_\mu e^{i \int d^4x [\mathcal{L}_0 + J^\mu A_\mu]}$$

where $J$ is an external source and

$$\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial^\mu A_\mu)^2 .$$

(a) Show that the action can be rewritten as

$$S[A^\mu] = \int d^4x \left[ \frac{1}{2} A_\mu K^{\mu\nu} A_\nu + J^\mu A_\mu \right]$$

with

$$K^{\mu\nu} = g^{\mu\nu} \Box - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu$$

where $\xi$ is an arbitrary gauge fixing parameter.

(b) If we define the Green’s function $G^{\mu\nu}$ by

$$\int d^4y \ K^{\mu\rho}(x - y) G_{\rho\sigma}(y - z) = g^{\mu\sigma} \delta^4(x - z)$$

with

$$K^{\mu\nu}(x - y) \equiv \delta^4(x - y) K^{\mu\nu}(x) ,$$

show that

$$G^{\mu\nu}(x - y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot (x-y)} \left[ -g^{\mu\nu} + (1 - \xi) \frac{k^\mu k^\nu}{k^2} \right] \frac{1}{k^2} .$$

N.B. If $\xi = 1$ it is the Feynman gauge, while if $\xi$ is chosen to be zero it becomes the Landau gauge.
PROBLEM 2: Interactions of Gauges Bosons with Fermions and Higgs Boson

The covariant derivative of the Standard Model of electroweak interactions is

\[ \mathcal{D}_\mu = \partial_\mu + igW^a_\mu t^a + ig'B_\mu \frac{Y}{2}, \quad a = 1, 2, 3, \]

where \( t^a \) and \( Y/2 \) are generators of \( SU(2)_L \times U(1)_Y \), \( e = g \sin \theta_W = g' \cos \theta_W \), and the charge operator is \( Q = t^3 + Y/2 \).

Let us consider

\[ \mathcal{L} = \bar{Q}_L i\gamma^\mu \mathcal{D}_\mu Q_L + \bar{u}_R i\gamma^\mu \mathcal{D}_\mu u_R + \bar{d}_R i\gamma^\mu \mathcal{D}_\mu d_R + (D_\mu \Phi)\dagger(D^\mu \Phi) \]

where \( a = 1, 2, 3 \),

\[ \psi_{L,R} = \frac{1}{2} (1 \mp \gamma_5) \psi, \]

and

\[ Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos \theta_W & \sin \theta_W \\ -\sin \theta_W & \cos \theta_W \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}, \quad \Phi = \begin{pmatrix} 0 \\ \frac{H + \nu}{\sqrt{2}} \end{pmatrix}, \]

in the unitary gauge with propagator of \( Z \) boson to be

\[ \Delta_F^{\mu\nu} = \frac{i [g^{\mu\nu} - (k^\mu k^\nu/M_Z^2)]}{k^2 - M_Z^2 + i\epsilon}, \quad \epsilon \to 0^+. \]

(a) Derive the relevant interaction Lagrangian involving \( W^3_\mu \) and \( B_\mu \) for \( u\bar{u} \to ZH \) with interactions of \( Zu\bar{u} \) and \( HZZ \).

(b) Draw Feynman diagrams and describe Feynman rules for \( Zu\bar{u} \) and \( HZZ \).

PROBLEM 3: Matrix Element of \( u\bar{u} \to ZH \)

Draw Feynman diagrams and evaluate the matrix element squared \( \langle |M|^2 \rangle \) for \( u(p_1)\bar{u}(p_2) \to Z(p_3)H(p_4) \), summed over all spins and polarizations as well as averaged over initial spins and colors.