PHYSICS 6433

Problem Set 2 – Due February 01, 2017

(1). Spin of the Electron

Let us define a generalized Pauli matrix (spin operator)

$$\Sigma_i \equiv \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \text{ with } \xi \equiv \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

such that

$$\alpha_i = \xi \Sigma_i = \Sigma_i \xi \ .$$

Thus we see that ξ commutes with Σ_i,

$$[\Sigma_i, \Sigma_j] = 2i\epsilon_{ijk}\Sigma_k \text{ and } [\Sigma_i, \beta] = 0 \ .$$

In terms of these matrices, we can write down the Dirac Hamiltonian as

$$H = c\vec{\alpha} \cdot \vec{P} + \beta mc^2 + V(r) = c\xi\Sigma_i \cdot P_i + \beta mc^2 + V(r) \ .$$

Here we have added a spherically symmetric potential to allow for interaction. We know that since the Hamiltonian is invariant under rotations, angular momentum must be conserved.

(a) Show that

$$[L_3, H] = i\hbar c\xi(\Sigma_1 P_2 - \Sigma_2 P_1) \ .$$

(b) Show that

$$[\Sigma_3, H] = 2ic\xi(\Sigma_2 P_1 - \Sigma_1 P_2) \ .$$

(c) Find the operator (J_3) for the total angular momentum that commutes with the Hamiltonian as a function of L_3 and Σ_3.

(d) Find the spin operator ($\vec{S} = \vec{J} - \vec{L}$) and the spin ($s$) for the electron.

Problems (2) and (3): Dirac Equation in an Electromagnetic Field

The classical interaction of a particle with charge Q can be expressed as

$$U(\vec{x}, \dot{\vec{x}}) = -\frac{Q}{c}\vec{v} \cdot \vec{A} + Q\Phi$$

from a 4-vector potential $A^\mu = (\Phi, \vec{A})$ where $\Phi =$ the electric potential and $\vec{A} =$ the magnetic vector potential. We can define a velocity operator $\vec{v} = c\vec{\alpha}$ in quantum mechanics for the Dirac equation. The the Hamiltonian becomes

$$H = c\vec{\alpha} \cdot (\vec{P} - \frac{Q}{c}\vec{A}) + \beta mc^2 + Q\Phi$$

and the Dirac equation becomes

$$[c\vec{\alpha} \cdot (\vec{P} - \frac{Q}{c}\vec{A}) + \beta mc^2 + Q\Phi]\psi = i\hbar\frac{\partial \psi}{\partial t}$$

In the non-relativistic limit, we have

$$E = E' + mc^2, \text{ with } E' \ll mc^2 .$$

(2). Let us consider the wave function of the form

$$\psi(\vec{x}, t) = \psi'(\vec{x}, t) e^{-(i/\hbar)mc^2 t} \text{ and } \psi' = \begin{pmatrix} \phi \\ \chi \end{pmatrix} .$$

Show that

$$i\hbar\frac{\partial \phi}{\partial t} = \left[\frac{1}{2m}\left(\vec{P} - \frac{Q}{c}\vec{A}\right)^2 - \frac{Q}{mc}\vec{S} \cdot \vec{B} + Q\Phi\right]\phi$$

where

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma} .$$

Hint: Let's define

$$\vec{\pi} = \vec{P} - \frac{Q}{c}\vec{A}$$

and rewrite the Dirac equation as

$$i\hbar\frac{\partial}{\partial t}\psi' = \left[c\vec{\alpha} \cdot \vec{\pi} + (\beta - I)mc^2 + Q\Phi\right]\psi'$$

with

$$(-2mc^2 + Q\Phi - i\hbar\frac{\partial}{\partial t})\chi \simeq -2mc^2\chi .$$

(3). In addition, let us consider an electron in a system with

- a weak electric field, $|Q\Phi| \ll mc^2$, and
- a uniform weak magnetic field $\vec{B}$ with $\vec{A} = (1/2)\vec{B} \times \vec{r}$.

Let us choose the Coulomb gauge $\nabla \cdot \vec{A} = 0$.

(a) Apply the tensor notation to show that

$$(\nabla \times \vec{A})_i = B_i \ .$$

(b) For a weak magnetic field, terms with $\vec{A}^2/c^2$ are negligible. Show that the equation of motion for ϕ becomes

$$i\hbar \frac{\partial \phi}{\partial t} = \left[\frac{\vec{P}^2}{2m} + \frac{e}{2mc} \vec{B} \cdot (\vec{L} + 2\vec{S}) - e\Phi \right] \phi$$

where $Q = -e =$ charge of the electron, $\vec{S} =$ the spin angular momentum, and $\vec{L} =$ the orbital angular momentum.