

Physics 5403

Problem Set 1 – Due September 4, 2009

(1). Noether's Theorem

For every continuous symmetry of the form $q_i \rightarrow z_i(q, \alpha)$, such that $L(z, \dot{z}) = L(q, \dot{q})$, there is a conservation law for the conserved quantity

$$Q = \sum_i \Lambda_i \frac{\partial L}{\partial q_i} = \text{constant},$$

where $\alpha = \text{constant}$,

$$\Lambda_i \equiv \left. \frac{\partial z_i(q, \alpha)}{\partial \alpha} \right|_{\alpha=0}, \quad \text{and} \quad \dot{\Lambda}_i = \left. \frac{\partial \dot{z}_i}{\partial \alpha} \right|_{\alpha=0}.$$

- For a rotation $\theta \rightarrow \phi = \theta + \alpha$ with a constant α , find Λ and $\dot{\Lambda}$.
- Find the conserved quantity Q associated with rotational invariance under a transformation $\phi = \theta + \alpha$ for the Lagrangian $L(r, \dot{r}, \theta, \dot{\theta})$ with a central force.

(2). Under an infinitesimal rotation about the z -axis with angle $\delta\phi$, the basis vector $|\phi\rangle$ and the state vector $|\psi\rangle$ become

$$\begin{aligned} U[R]|\phi\rangle &= |\phi + \delta\phi\rangle, \quad \text{and} \\ U[R]|\psi\rangle &= |\psi'\rangle \end{aligned}$$

where $U[R]$ is the rotation operator in the Hilbert space and R is the rotation operator in the coordinate space.

In the ϕ -basis, let us define

$$\psi(\phi) \equiv \langle \phi | \psi \rangle$$

and consider

$$\psi'(\phi) \equiv \langle \phi | \psi' \rangle = \langle \phi | U[R(\delta\phi)] | \psi \rangle = R(\delta\phi)\psi(\phi)$$

for a spherically symmetric system.

- Find the rotation operator $R(\delta\phi)$.
- Express $R(\delta\phi)$ in terms of the angular momentum

$$L_z = -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) = -i\hbar \frac{\partial}{\partial \phi}.$$

- Describe the condition for rotational invariance and identify the conserved quantity for this symmetry.
- Find the rotation operator $R(\Delta\phi)$ for a finite rotation ($\Delta\phi$).

(3). For a system with a spherically symmetric potential, the complete solution to the Schrödinger equation is

$$\psi_{\ell,m}(r, \theta, \phi) = R(r)Y_{\ell,m}(\theta, \phi)$$

where

$$Y_{\ell,m} = \epsilon \sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - |m|)!}{(\ell + |m|)!}} P_{\ell,m}(\cos \theta) e^{im\phi}$$

with $\epsilon = (-1)^m$ for $m > 0$ and $\epsilon = +1$ for $m < 0$,

$$P_{\ell,m}(x) = (1 - x^2)^{\frac{|m|}{2}} \frac{d^{|m|}}{dx^{|m|}} P_{\ell}(x)$$

with $x = \cos \theta$ and

$$P_{\ell}(x) = \frac{1}{2^{\ell} \ell!} \frac{d^{\ell}}{dx^{\ell}} (x^2 - 1)^{\ell}.$$

Parity means reflecting a vector through the origin. In spherical coordinates, a position vector is described with $\vec{r} = (r, \theta, \phi)$.

- (a) Find $\Pi(r, \theta, \phi)$.
- (b) Find $\Pi e^{im\phi}$.
- (c) Find $\Pi P_{\ell,m}(x)$.
- (d) Find $\Pi Y_{\ell,m}(\theta, \phi)$.
- (e) Find $\Pi R(r)$.

where $\Pi =$ the parity operator.