PHYS 6213: Advanced Particle Physics

Problem Set 1 – due February 02

PROBLEM 1: Gauge Transformation in QED

The interaction Lagrangian density of the quantum electrodynamics for the electron $[\psi(x)]$ and the photon $[A_\mu(x)]$ is

$$\mathcal{L}_{\mathrm{I}} = \bar{\psi}(x) i\gamma^\mu D_\mu \psi(x)$$

where $\bar{\psi} \equiv \psi^\dagger \gamma^0$, γ's are the Dirac gamma matrices, the covariant derivative is $D_\mu = \partial_\mu + ieA_\mu Q$, e is the charge of the proton, and Q is the Hermitian charge operator that does not depend on space and time. Let us consider the local U(1) gauge transformation

$$\psi'(x) = e^{-i\theta(x)Q}\psi(x) \ .$$

(a) Apply the eigenvalue equation of Q for an electron field, find $\psi'(x)$ and $\bar{\psi}'$, then find $\Delta\mathcal{L}$ for $\mathcal{L}' = \mathcal{L} + \Delta\mathcal{L}$ under the U(1) gauge transformation, where L is the original Lagrangian.

(c) Find A'_μ under the U(1) gauge transformation such that the Lagrangian density is invariant.

PROBLEM 2: The Adjoint Representation of SU(2)

Let us consider three 3×3 matrices G_i with elements given by

$$(G_i)_{jk} = -i\hbar\epsilon_{ijk} \quad i, j, k = 1, 2, 3$$

where j and k are the row and column indices.

(a) Show that G_i's satisfy the angular momentum commutation relations: $[G_i, G_j] = i\hbar\epsilon_{ijk}G_k$.

(b) Find the matrix G_3, then calculate the eigenvalues λ_i and normalized eigenvectors $|\lambda_i\rangle$ of G_3.

(c) Find a unitary matrix that transforms G_3 to J_3 for $j = 1$ with J_3 being diagonal

$$J_3 = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

N.B. This unitary matrix transforms the Cartesian space representation of the angular momentum operator G_i to its spherical basis representation J_i for $j = 1$. This problem is helpful in understanding the spin of photon.

PROBLEM 3: The Conjugate Representation of SU(2)

For a doublet (ψ) of two particles with isospin$-1/2$, the SU(2) transformation is often expressed as

$$\psi' = U\psi, \;\; \psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}, \;\; U(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{t}}$$

where $\vec{t} = \vec{\sigma}/2$, α's are the real group parameters, and σ's are the Pauli matrices.

(a) For every 2×2 unitary matrix U with unit determinant, find a real matrix S which connects U to its complex conjugate matrix U^* through the similarity transformation $S^{-1}US = U^*$. *Hint: It is easier to consider an infinitesimal transformation* $U(\vec{\epsilon}) = e^{i\vec{\epsilon}\cdot\vec{t}}$, *where* $\vec{\epsilon}$ *is a constant infinitesimal vector.*

(b) The conjugate representation of SU(2) transforms as

$$\psi'^* = U^*\psi^* \, .$$

Show that

$$(S\psi'^*) = U(S\psi^*).$$

This means that $S\psi^*$ has the same transformation properties as ψ. Thus we can define

$$\phi^a \;\; \equiv \;\; \epsilon^{ab}\phi_b = \epsilon^{ab}(\psi^b)^* = (S\psi^*)^a,$$

where $\epsilon^{12} = 1, \epsilon^{21} = -1, \epsilon^{11} = \epsilon^{22} = 0$, such that $\phi' = U\phi$.

(c) Let us choose ψ^1 and ψ^2 to be eigenvectors for the isospin$-1/2$ representation or the fundamental representation of SU(2), with eigenvalues $\pm 1/2$ for the diagonal generator t_3,

$$t_3\psi^1 = \frac{1}{2}\psi^1, \;\; t_3\psi^2 = -\frac{1}{2}\psi^2, \;\; \psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$$

Find the eigenvalues of t_3 operating on ψ^{1*} and ψ^{2^*}.

(d) Find the isospin doublet $\bar{N}$ in the conjugate representation with $\bar{p}$ and $\bar{n}$ and construct the isospin states for a composite system of a nucleon-antinucleon pair $(N\bar{N})$, with $I = I_N + I_{\bar{N}}$, and

$$N = \begin{pmatrix} p \\ n \end{pmatrix} \; .$$