PHYS 6433: Quantum Field Theory

Problem Set 1 – Due January 25, 2017

(1). Simple Harmonic Oscillator

A block of mass m moves on a frictionless surface with a displacement x under the influence of a spring force

$$F = -kx,$$

where k is the spring constant.

(a) Calculate the potential energy $U(x)$, and find the Lagrangian.

(b) Find the Euler-Lagrange equation.

(c) Find the Hamiltonian.

(d) Find the Hamilton's equations.

(2). Spherically Symmetric System

A particle of mass m moves on a plane under the influence of a central force

$$\vec{F} = -\frac{\alpha}{r^2}\hat{r}, \quad \alpha > 0.$$

where $\hat{r}$ is the unit vector in the radial direction.

(a) Calculate the potential energy $U(r)$, and find the Lagrangian in terms of polar coordinates r and θ.

(b) Find the Euler-Lagrange equations, then identify the cyclic variables and conserved quantities.

(c) Find the Hamiltonian.

(d) Find the Hamilton's equations.

(3). Hamilton's Principle

The action of a mechanical system with N generalized coordinates is defined as

$$S[q_i] = \int L(q_i, \dot{q}_i) dt$$

where $i = 1, \cdots, N$. The Lagrangian L is defined as

$$L(q_i, \dot{q}_i) = T(q_i, \dot{q}_i) - U(q_i)$$

where $T =$ the kinetic energy and $U =$ the potential energy.
Applying the Hamilton's Principle to show that the Euler-Lagrange equations have the following form

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0,$$

for $i = 1, \cdots, N$.

(4). Dirac Matrices

To combine quantum mechanics with special relativity, Dirac suggested a linear relationship between the Hamiltonian (H) and the momentum ($\vec{P}$)

$$H = c\vec{\alpha} \cdot \vec{P} + \beta mc^2$$

where α_i and β are $N \times N$ matrices.

If H is the correct Hamiltonian, by squaring it we should get back the relation from relativity. That is

$$\begin{aligned} c^2\vec{P}^2 + m^2c^4 &= H^2 \\ &= c^2\left[\frac{1}{2}\{\alpha_i, \alpha_j\}P_iP_j + \{\alpha_i, \beta\}mcP_i + \beta^2m^2c^2\right] . \end{aligned}$$

Thus it is clear that the left hand side equals the right hand side if

$$\begin{aligned} \beta^2 &= I \\ \alpha_i^2 &= I \end{aligned}$$

and

$$\begin{aligned} \{\alpha_i, \beta\} &= 0 \\ \{\alpha_i, \alpha_j\} &= 0 \quad \text{for} \;\; i \neq j \end{aligned}$$

or

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \; .$$

For $N = 4$ let us choose β to be diagonal and traceless.

$$\begin{aligned} \beta &= \begin{pmatrix} I & O \\ O & -I \end{pmatrix} , \\ I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \\ O &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} . \end{aligned}$$

In addition, let's choose

$$\alpha_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix},$$

where A_i, B_i, C_i and D_i are 2×2 matrices.

Apply $\{\alpha_i, \beta\} = 0$, $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$, $\alpha_i^\dagger = \alpha_i$, and show that

$$\text{(a)} \;\; \alpha_i = \begin{pmatrix} 0 & B_i \\ B_i^\dagger & 0 \end{pmatrix} ,$$

and

$$\text{(b)} \;\; Bi^\dagger B_j + B_j^\dagger B_i = 0 \; .$$

We can choose $B_i = \sigma_i$ with

$$\begin{aligned} \sigma_i^\dagger &= \sigma_i \; , \quad \text{and} \\ \sigma_i \sigma_j + \sigma_j \sigma_i &= 2\delta ij \; , \end{aligned}$$

then α_i's become

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} .$$