(1). Simple Harmonic Oscillator

A block of mass $m$ moves on a frictionless surface with a displacement $x$ under the influence of a spring force

$$F = -kx,$$

where $k$ is the spring constant.

(a) Calculate the potential energy $U(x)$, and find the Lagrangian.

(b) Find the Euler-Lagrange equation.

(c) Find the Hamiltonian.

(d) Find the Hamilton’s equations.

(2). Spherically Symmetric System

A particle of mass $m$ moves on a plane under the influence of a central force

$$\vec{F} = -\frac{\alpha}{r^2} \hat{r}, \quad \alpha > 0,$$

where $\hat{r}$ is the unit vector in the radial direction.

(a) Calculate the potential energy $U(r)$, and find the Lagrangian in terms of polar coordinates $r$ and $\theta$.

(b) Find the Euler-Lagrange equations, then identify the cyclic variables and conserved quantities.

(c) Find the Hamiltonian.

(d) Find the Hamilton’s equations.
(3). Hamilton’s Principle

The action of a mechanical system with \( N \) generalized coordinates is defined as

\[
S[q_i] = \int L(q_i, \dot{q}_i)dt
\]

where \( i = 1, \cdots, N \). The Lagrangian \( L \) is defined as

\[
L(q_i, \dot{q}_i) = T(q_i, \dot{q}_i) - U(q_i)
\]

where \( T = \) the kinetic energy and \( U = \) the potential energy.

Applying the Hamilton’s Principle to show that the Euler-Lagrange equations have the following form

\[
\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = 0,
\]

for \( i = 1, \cdots, N \).

(4). Dirac Matrices

To combine quantum mechanics with special relativity, Dirac suggested a linear relationship between the Hamiltonian (\( H \)) and the momentum (\( \vec{P} \))

\[
H = c\bar{\alpha} \cdot \vec{P} + \beta mc^2
\]

where \( \alpha_i \) and \( \beta \) are \( N \times N \) matrices.

If \( H \) is the correct Hamiltonian, by squaring it we should get back the relation from relativity. That is

\[
c^2\vec{P}^2 + m^2c^4 = H^2 = c^2 \left[ \frac{1}{2}\{\alpha_i, \alpha_j\}P_iP_j + \{\alpha_i, \beta\}mcP_i + \beta^2m^2c^2 \right].
\]

Thus it is clear that the left hand side equals the right hand side if

\[
\beta^2 = I
\]

\[
\alpha_i^2 = I
\]

and

\[
\{\alpha_i, \beta\} = 0
\]

\[
\{\alpha_i, \alpha_j\} = 0 \quad \text{for} \quad i \neq j
\]

or

\[
\{\alpha_i, \alpha_j\} = 2\delta_{ij}.
\]
For $N = 4$ let us choose $\beta$ to be diagonal and traceless.

\[
\beta = \begin{pmatrix} I & O \\ O & -I \end{pmatrix},
\]
\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and}
\]
\[
O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

In addition, let’s choose

\[
\alpha_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix},
\]

where $A_i, B_i, C_i$ and $D_i$ are $2 \times 2$ matrices.

Apply $\{\alpha_i,\beta\} = 0$, $\{\alpha_i,\alpha_j\} = 2\delta_{ij}$, $\alpha_i^\dagger = \alpha_i$, and show that

(a) $\alpha_i = \begin{pmatrix} 0 & B_i \\ B_i^\dagger & 0 \end{pmatrix},$

and

(b) $B_i^\dagger B_j + B_j^\dagger B_i = 0$.

We can choose $B_i = \sigma_i$ with

\[
\sigma_i^\dagger = \sigma_i, \text{ and}
\]
\[
\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij},
\]

then $\alpha_i$’s become

\[
\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}.
\]