PHYS 5393: Quantum Mechanics I
Problem Set 12–Due December 04, 2019

Problem (1)

A particle moves in an infinite square well of width $2a$ with the potential energy

$$V(x) = \begin{cases} 0, & \text{for } -a < x < a \text{ with } a > 0, \text{ and} \\ \infty, & \text{otherwise.} \end{cases}$$

(a) Solve the Schrödinger equation and show that the normalized first excited state wave function $u_1(x)$ inside the well is

$$u_1(x) = \frac{1}{\sqrt{a}} \sin\left(\frac{\pi x}{a}\right).$$

At time $t = 0$, a normalized wave function inside the well is

$$\psi(x) = \Psi(x, 0) = C\left[5 \cos \frac{\pi x}{2a} + 12 \sin \frac{\pi x}{a}\right].$$

(b) Find the normalization constant C.

(c) What is the wave function $\Psi(x, t)$?

(d) If a measurement of total energy is made, what are the possible results of such a measurement, and what is the probability to measure each of them?

(e) What is the quantum expectation value of the energy $\langle E \rangle$?

Problem (2)

For a system with a spherically symmetric potential, the complete solution to the Schrödinger equation is

$$\psi_{\ell,m}(r, \theta, \phi) = R(r) Y_{\ell,m}(\theta, \phi)$$

where

$$Y_{\ell,m} = \epsilon \sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - |m|)!}{(\ell + |m|)!}} P_{\ell,m}(\cos\theta) e^{im\phi}$$

with $\epsilon = (-1)^m$ for $m > 0$ and $\epsilon = +1$ for $m < 0$,

$$P_{\ell,m}(z) = (1 - z^2)^{\frac{|m|}{2}} \frac{d^{|m|}}{dz^{|m|}} P_\ell(z)$$

with $z = \cos\theta$ and

$$P_\ell(z) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dz^\ell} (z^2 - 1)^\ell.$$

Parity means reflecting a vector through the origin. In spherical coordinates, a position vector is described with $\vec{r} = (r, \theta, \phi)$. Find (a) $\Pi e^{im\phi}$, (b) $\Pi P_{\ell,m}(z)$, and (c) $\Pi Y_{\ell,m}(\theta, \phi)$, where $\Pi \equiv$ the parity operator.

3. **Spin 1/2 systems**

Consider a spin half particle. The basis states for describing the wave function are denoted by $|+\rangle$ and $|-\rangle$ which are the spin up and spin down eigenstates of the spin along the $\hat{z}-$direction.

You build an apparatus that can measure the spin of the particle in any arbitrary direction $\hat{z'}$, where $\hat{z'}$ is a unit vector given by the usual polar and azimuthal angles $\hat{z'} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$.

(a) If your apparatus were to measure the spin along this arbitrary direction $\hat{z'}$, what range of values in units of $\hbar$ could you possibly obtain? Please provide an explicit derivation. (2 points)

(b) Suppose your apparatus now actually measures the spin along the $\hat{z'}$ direction and gives you an answer $+\hbar/2$. This act of measurement changes the state of the particle, putting it into a state $|\psi\rangle$. Express $|\psi\rangle$ in terms of $|+\rangle$ and $|-\rangle$. (6 points)

(c) Now the particle is in state $|\psi\rangle$. You ask your apparatus to measure the spin in the original $\hat{z}$ direction. What is the probability that the answer will be $+\hbar/2$? (2 points)