Physics 5393

Problem Set 11–Due April 27, 2011

Problem (1)

Starting from the Rodrigues formula, derive the orthonormal condition for Legendre polynomials:

$$\int_{-1}^{1} P_\ell(z) P_{\ell'}(z)\, dz \;=\; \left(\frac{2}{2\ell+1}\right)\delta_{\ell\ell'}\,.$$

Problem (2): Quantum Virial Theorem

Let us consider a Hermitian operator Ω and a normalized state vector $\psi\rangle$ for a bound state with a definite energy:

$$H|\psi\rangle = E|\psi\rangle$$

where the Hamiltonian is

$$H = \frac{P^2}{2m} + V(R)\,.$$

(a) Find the expectation value of the commutator $[\Omega, H]$ in the state $|\psi\rangle$.

(b) Show that

$$[\vec{R}, H] = i\hbar\frac{\vec{P}}{m} \quad \text{and} \quad [\vec{P}, H] = -i\hbar\nabla V(R)\,.$$

(c) Apply the results in parts (a) and (b) with $\Omega = \vec{R}\cdot\vec{P}$, and show prove that

$$2\langle\psi|T|\psi\rangle = \langle\psi|R\frac{\partial V}{\partial R}|\psi\rangle$$

where T is the kinetic energy and $|\psi\rangle$ is an energy eigenstate. This is the quantum version of the virial theorem.

(d) If $V(R)$ is proportional to R^m, show that for an energy eigenstate with energy E,

$$E = \left(1 + \frac{m}{2}\right)\langle\psi|V|\psi\rangle\,.$$

Problems 3 and 4

For the hydrogen atom, the ground state wave function is

$$\psi(r) = \psi_{100}(r) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a} \,,$$

and the wave functions of the hydrogen atom with $n = 2, \ell = 1, m = \pm 1$ are

$$\psi_{21\pm1}(\vec{r}) = \mp \frac{1}{\sqrt{\pi a}} \frac{1}{8a^2} \, r \, e^{-r/2a} \sin\theta \, e^{\pm i\phi} \,.$$

Problem (3)

(a) Find $\langle R \rangle$ and $\langle R^2 \rangle$ for an electron in the ground state of hydrogen. Express your answers in terms of the Bohr radius.

(b) Find $\langle X \rangle$ and $\langle X^2 \rangle$ for an electron in the ground state of hydrogen atom. *Hint:* This requires no new integration–note that $R^2 = X^2 + Y^2 + Z^2$, and exploit the symmetry of the ground state.

(c) Find $\langle X^2 \rangle$ in the state $n = 2, \ell = 1, m = 1$ with $X = R \sin\theta \cos\phi$.

Problem (4)

A hydrogen atom starts out in the following linear combination of the stationary states $n = 2, \ell = 1, m = 1$ and $n = 2, \ell = 1, m = -1$

$$\Psi(\vec{r}, 0) = \frac{1}{\sqrt{2}} (\psi_{211} + \psi_{21-1}) \ .$$

(a) Construct $\Psi(\vec{r}, t)$, Simplify it as much as you can.

(b) Find the expectation value of the potential energy, $\langle V \rangle$. Give both the formula and the actual number in electron volts.