PHYS 3803: Quantum Mechanics I
Problem Set 10–Due April 30, 2021

Problems 1 − 3

Let us choose $|\ell, m\rangle$ as the common eigenvector for angular momentum operators L^2 and $L_z = L_3$ such that

$$\begin{aligned} L_z|\ell, m\rangle &= \hbar m|\ell, m\rangle\,, \quad \text{and} \\ L^2|\ell, m\rangle &= \hbar^2\ell(\ell+1)|\ell, m\rangle \end{aligned}$$

where m is a quantum number $-\ell \le m \le \ell$. The eigenvectors $|\ell, m\rangle$ form a complete set of orthonormal basis with $\langle \ell', m'|\ell, m\rangle = \delta_{\ell',\ell}\delta_{m',m}$

In addition, the lowering and raising operators are defined as

$$\begin{aligned} L_- &\equiv L_1 - iL_2 = L_x - iL_y \\ L_+ &\equiv L_1 + iL_2 = L_x + iL_y\,. \end{aligned}$$

Problem (1)

Find the coefficients c_m and d_m for

$$\begin{aligned} L_-|\ell, m\rangle &= c_m\,|\ell, m-1\rangle \\ L_+|\ell, m\rangle &= d_m\,|\ell, m+1\rangle \end{aligned}$$

where $L^2 = L_iL_i = L_1^2 + L_2^2 + L_3^2$.

Problem (2)

The angular momentum operators have interesting relations

$$\begin{aligned} L_-|\ell, m\rangle &= \hbar\sqrt{\ell(\ell+1) - m(m-1)}|\ell, m-1\rangle\,, \\ L_+|\ell, m\rangle &= \hbar\sqrt{\ell(\ell+1) - m(m+1)}|\ell, m+1\rangle\,. \end{aligned}$$

Calculate $\langle L_x\rangle$, $\langle L_y\rangle$, $\langle L_x^2\rangle$ and $\langle L_y^2\rangle$ in such a state denoted by $|\ell, m\rangle$.

Problem (3)

The eigenvalue of L_z has a maximum value $m_{\text{MAX}} = \ell$ such that $L_+|\ell, \ell\rangle = 0$. In the spherical coordinate basis, we have

$$\begin{aligned} L_z &= -i\hbar\frac{\partial}{\partial\phi}\,, \quad \text{and} \\ L_+ &= \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\,. \end{aligned}$$

Let us define

$$
\begin{aligned}
U_{\ell,m} &\equiv \langle r,\theta,\phi|\ell,m\rangle \quad \text{and} \\
U_{\ell,\ell}(r,\theta,\phi) &= R_{\ell,\ell}(r)\Theta_{\ell,\ell}(\theta)\Phi_{\ell}(\phi)\,.
\end{aligned}
$$

Apply separation of variables and show that

$$
\begin{aligned}
\Phi_{\ell}(\phi) &= A_{\phi}e^{i\ell\phi}\,, \\
\Theta_{\ell,\ell}(\theta) &= A_{\theta}(\sin\theta)^{\ell}\,,
\end{aligned}
$$

and

$$U_{\ell,\ell}(r,\theta,\phi) = AR_{\ell,\ell}(\sin\theta)^{\ell}e^{i\ell\phi}$$

where A_{ϕ}, A_{θ} and A are normalization constants.

Problems 4 and 5

For the hydrogen atom, the ground state wave function is

$$\psi(r) = \psi_{100}(r) = \frac{1}{\sqrt{\pi a^3}}e^{-r/a}\,,$$

and the wave functions of the hydrogen atom with $n=2, \ell=1, m=\pm 1$ are

$$\psi_{21\pm1}(\vec{r}) = \mp\frac{1}{\sqrt{\pi a}}\frac{1}{8a^2}\,r\,e^{-r/2a}\sin\theta\,e^{\pm i\phi}\,.$$

Problem (4) [Griffiths 4.13]

(a) Find $\langle R\rangle$ and $\langle R^2\rangle$ for an electron in the ground state of hydrogen. Express your answers in terms of the Bohr radius.

(b) Find $\langle X\rangle$ and $\langle X^2\rangle$ for an electron in the ground state of hydrogen atom. *Hint:* This requires no new integration–note that $R^2 = X^2+Y^2+Z^2$, and exploit the symmetry of the ground state.

(c) Find $\langle X^2\rangle$ in the state $n=2, \ell=1, m=1$ with $X = R\sin\theta\cos\phi$.

Problem (5) [Griffiths 4.15]

A hydrogen atom starts out in the following linear combination of the stationary states $n=2, \ell=1, m=1$ and $n=2, \ell=1, m=-1$

$$\Psi(\vec{r},0) = \frac{1}{\sqrt{2}}(\psi_{211}+\psi 21-1)\ .$$

(a) Construct $\Psi(\vec{r},t)$, Simplify it as much as you can.

(b) Find the expectation value of the potential energy, $\langle V\rangle$. Give both the formula and the actual number in electron volts.