PHYS 5213/4213: Nuclear and Particle Physics, Autumn 2021

# Problem Set 10 – due December 03

**Problem (1): Finite Rotation for a Dirac Spinor**

Let us consider a finite rotation of angle $\phi$ about the $z-$axis with $x' = \Lambda x$ and

$$\omega_{12} = \epsilon_{123}\phi_3 = \phi_3 = \phi \quad \text{where} \quad \omega_{ij} = \epsilon_{ijk}\phi_k$$

Then the Dirac spinor will change under this rotation as $\psi'(x') = S_R(\Lambda)\psi(x)$.

(a) Find the rotation operator $\Lambda$ as a $4 \times 4$ matrix.

(b) Find $\sigma_{ij}$ for an infinitesimal rotation

$$S_R(\Lambda) = I - \frac{i}{4}\sigma_{ij}\omega^{ij} .$$

(c) Show that for a finite rotation of angle $\phi$ about the $z-$axis,

$$S_R(\Lambda) = e^{-\frac{i}{2}\Sigma_3\phi_3} = I\cos\left(\frac{\phi}{2}\right) - i\Sigma_3\sin\left(\frac{\phi}{2}\right) \quad \text{where} \quad \Sigma_3 = \sigma_{12} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} .$$

(d) Find the rotation operator $S_R(\Lambda)$ as a $4 \times 4$ matrix for a Dirac spinor.

**Problem (2): The Hamiltonian of a Scalar Field**

In the case of free Klein-Gordon theory with quantized fields, the Hamiltonian is

$$\begin{aligned} H &= \int \mathcal{H}\, d^3x \\ &= \int \left(\frac{1}{2}\pi^2(x) + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{1}{2}m^2\phi^2\right) d^3x \end{aligned}$$

where

$$\begin{aligned} \phi(x) &= \int \frac{d^3k}{(2\pi)^3\sqrt{2k^0}}\left[a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right] \quad \text{and} \\ \pi(x) &= -i\int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{k^0}{2}}\left[a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right] . \end{aligned}$$

The operators $a(\vec{k})$ and $a^\dagger(\vec{k})$ have commutation relations analogous to the annihilation and creation operators of a harmonic oscillator:

$$[a(\vec{k}), a(\vec{q})] = 0 = [a^\dagger(\vec{k}), a^\dagger(\vec{q})] \;\; \text{and} \;\; [a(\vec{k}), a^\dagger(\vec{q})] = (2\pi)^3\delta^3(k-q) .$$

(a) Show that

$$\int \pi^2(x)\, d^3x = -\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}(k^0)[-a(\vec{k})a^\dagger(\vec{k}) - a^\dagger(\vec{k})a(\vec{k}) + e^{-2ik^0x^0}a(\vec{k})a(-\vec{k}) + e^{2ik^0x^0}a^\dagger(\vec{k})a^\dagger(-\vec{k})]\,.$$

(b) Show that

$$\int \nabla\phi(x)\cdot\nabla\phi(x)\, d^3x = -\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}(\frac{|\vec{k}|^2}{k^0})[-a(\vec{k})a^\dagger(\vec{k}) - a^\dagger(\vec{k})a(\vec{k}) - e^{-2ik^0x^0}a(\vec{k})a(-\vec{k}) - e^{2ik^0x^0}a^\dagger(\vec{k})a^\dagger(-\vec{k})]\,.$$

(c) Show that

$$\int \phi^2(x)\, d^3x = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^0}[a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k}) + e^{-2ik^0x^0}a(\vec{k})a(-\vec{k}) + e^{2ik^0x^0}a^\dagger(\vec{k})a^\dagger(-\vec{k})]\,.$$

(d) Substitute (a)-(c) into the Hamiltonian, and show that

$$H = \int \frac{d^3k}{(2\pi)^3}\frac{\omega_k}{2}\left[a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})\right]\,.$$

**Problem (3): Scattering in Quantum Electrodynamics**

The interaction Lagrangian of QED for $e^-e^+ \to \gamma \to \mu^-\mu^+$ is

$$\mathcal{L}_{\text{QED}} = e\bar{\psi}_e\gamma^\mu A_\mu\psi_e + e\bar{\psi}_\mu\gamma^\mu A_\mu\psi_\mu\,.$$

(a) Draw Feynman diagrams for the photon propagator and the $e^-e^+\gamma$ vertex and describe the Feynman Rules.

(b) Show that the spin averaged amplitude squared is

$$\langle|M|^2\rangle[e^-(p_1)e^+(p_2) \to \mu^-(p_3)\mu^-(p_4)] = 2e^4\left(\frac{t^2+u^2}{s^2}\right)$$

at high energy limit ($m_e \sim 0 \sim m_\mu$) in terms of Mandelstam variables $s = (p_1+p_2)^2$, $t = (p_2-p_3)^2$, and $u = (p_1-p_3)^2$.

(c) Show that in the CM frame, the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right)$$

with $\phi = 0$ and $e^2 = 4\pi\alpha$.

(d) Integrate over $\theta$, $\phi$ and find the cross section $\sigma(e^-e^+ \to \mu^-\mu^+)$ in the CM frame.