PHYS 3803: Quantum Mechanics I, Spring 2021

Lecture 25, April 29, 2021 (Thursday)

- Reading: Hydrogen Atom—Griffiths 4.2
- Assignments: Problem Set 10 due May 03 (Monday).
  Submit your homework assignments to Canvas.

Topics for Today: Hydrogen Atom [Griffiths 4.2]

6.2 Introduction to the Hydrogen Atom

6.3 Fundamental Quantities Associated with Hydrogen Atom

Next Lecture: Hydrogen Atom [Griffiths 4.2]

6.3 Fundamental Quantities Associated with Hydrogen Atom

6.4 Numerical Estimates

6.2 Hydrogen Atom

Defining $R_{E,\ell} \equiv u_{E,\ell}(r)/r$, we can write the equation for $u_{E,\ell}$ as

$$\frac{d^2 u_{E,\ell}}{dr^2} + \frac{2\mu}{\hbar^2}\left[E + \frac{e^2}{r} - \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\right] u_{E,\ell} = 0\,.$$

Let us introduce a dimensionless parameter

$$y = 2\left(\frac{2\mu|E|}{\hbar^2}\right)^{1/2} r \quad \text{and} \quad \frac{d}{dr} = 2\left(\frac{2\mu|E|}{\hbar^2}\right)^{1/2} \frac{d}{dy}\,.$$

In terms of the y variables, the equation now becomes

$$\frac{d^2 u_{E,\ell}}{dy^2} + \left[-\frac{1}{4} + \frac{\lambda}{y} - \frac{\ell(\ell+1)}{y^2}\right] u_{E,\ell} = 0\,,$$

where

$$\lambda = \left(\frac{\mu}{2\hbar^2|E|}\right)^{1/2} e^2 = \left(\frac{\mu e^4}{2\hbar^2|E|}\right)^{1/2}\,.$$

This suggests a general solution of the form

$$u_{E,\ell}(y) = e^{-\frac{1}{2}y} v(y) = e^{-\frac{1}{2}y} y^{\ell+1} \sum_{k=0}^{\infty} a_k y^k \, .$$

Thus the equation for v becomes

$$\frac{d^2 v}{dy^2} - \frac{dv}{dy} + \left[\frac{\lambda}{y} - \frac{\ell(\ell+1)}{y^2} \right] v = 0 \, .$$

Applying the power series expansion of $v(y)$

$$v(y) = \sum_{k=0}^{\infty} a_k y^{k+\ell+1}$$

we obtain

$$\sum_{k=0}^{\infty} \left[\Big((k+\ell+1)(k+\ell) - \ell(\ell+1) \Big) a_k y^{k+\ell-1} - (k+\ell+1-\lambda) a_k y^{k+\ell} \right] = 0$$

In general the recursion relation would connect a_{k+1} to a_k. Thus looking at the coefficient of $y^{k+\ell}$ we have

$$[(k+1+\ell+1)(k+1+\ell)-\ell(\ell+1)]a_{k+1}=(k+\ell+1-\lambda)a_k \quad \text{or}$$

$$a_{k+1}=\frac{(k+\ell+1-\lambda)}{(k+1)(k+2\ell+2)}a_k\,.$$

Thus for large k

$$\frac{a_{k+1}}{a_k}\to\frac{1}{k}\,.$$

Unless the series terminates this would lead to an unphysical solution. The series terminates if

$$k+\ell+1-\lambda=0\,.$$

That leads to

$$\lambda=\left(\frac{\mu e^4}{2\hbar|E|}\right)^{1/2}=k+\ell+1=n \quad \text{and} \quad E_n=-\frac{\mu e^4}{2n^2\hbar^2}\,.$$

Since both k and ℓ take positive integer values, n also take positive integer values. Even when ℓ and k are both equal to zero, $n = 1$. Thus the allowed values for n are

$$n = 1, 2, 3, \cdots, \infty$$

In addition,

$$\ell = n - k - 1 = n - 1, n - 2, \cdots, 0$$

These are the allowed values of the orbital angular momentum for a given value of n.

Thus the solution of the differential equation is

$$\begin{aligned} u_{n,\ell}(y) &= e^{-\frac{1}{2}y} \sum_{k=0}^{n-\ell-1} a_k y^{k+\ell+1} \\ &= e^{-\frac{1}{2}y} y^{\ell+1} w(y) \end{aligned}$$

The differential equation that $w(y)$ satisfies is

$$y\frac{d^2w}{dy^2} + (2\ell + 2 - y)\frac{dw}{dy} + (n - \ell - 1)w(y) = 0$$

An equation of the form

$$y\frac{d^2L_q}{dy^2} + (1 - y)\frac{dL_q}{dy} + qL_q = 0$$

is called the Laguerre equation and $L_q^{\prime}s$ are called Laguerre polynomials of order q. The functions $L_q^p(y)$ are related to the L_q's by the relation

$$L_q^p(y) = \frac{d^p}{dy^p}L_q(y)\,, \quad q \geq p\,,$$

and they are known as the associated Laguerre polynomials. They are polynomials of order $q - p$ and satisfy the differential equation

$$y\frac{d^2L_q^p}{dy^2} + (p + 1 - y)\frac{dL_q^p}{dy} + (q - p)L_q^p = 0$$

Comparing this equation with the one satisfied by the w's, we see that

$$w(y) = L_{n+\ell}^{2\ell+1}(y) \; .$$

The radial wave function for the Hydrogen atom is

$$\begin{aligned} R_{n,\ell} &= N_{n,\ell} e^{-\frac{1}{2}y} y^{\ell} L_{n+\ell}^{2\ell+1}(y) \quad \text{where} \\ y &= 2\left(\frac{2\mu|E_n|}{\hbar^2}\right)^{1/2} r \;=\; \left(\frac{2\mu e^2}{\hbar^2 n}\right) r \, . \end{aligned}$$

And the total wave function is

$$\psi_{n,\ell,m}(r,\theta,\phi) = R_{n,\ell}(r) Y_{\ell,m}(\theta,\phi) \; .$$

N.B Griffiths chooses the radial wave function as

$$R_{n,\ell} = N_{n,\ell} e^{-\frac{1}{2}y} y^{\ell} L_{n-\ell-1}^{2\ell+1}(y) \, .$$

Laguerre polynomials

It is convenient to evaluate the Laguerre polynomials with

$$L_p(y) = e^y \frac{d^p}{dy^p} \left(y^p e^{-y} \right) \; .$$

Exercise: Find L_0, L_1, L_2 and L_3.

Associated Laguerre polynomials

The associated Laguerre polynomials are defined as

$$L_p^q(y) = \frac{d^q}{dy^q} L_p(y) \, .$$

Exercise: Find L_1^1, L_2^1, and L_3^3.

Laguerre polynomials

It is convenient to evaluate the Laguerre polynomials with

$$L_p(y) = e^y \frac{d^p}{dy^p} \left(y^p e^{-y} \right) \ .$$

For example,

$$\begin{aligned}
L_0(y) &= e^y \frac{d^0}{dy^0} \left(y^0 e^{-y} \right) = 1 \, , \\
L_1(y) &= e^y \frac{d}{dy} \left(y e^{-y} \right) = -y + 1 \, , \\
L_2(y) &= e^y \frac{d^2}{dy^2} \left(y^2 e^{-y} \right) = y^2 - 4y + 2 \, , \\
L_3(y) &= e^y \frac{d^3}{dy^3} \left(y^3 e^{-y} \right) = -y^3 + 9y^2 - 18y + 6 \, .
\end{aligned}$$

Associated Laguerre polynomials

The radial wave function for the Hydrogen atom is

$$R_{n,\ell} \quad = \quad N_{n,\ell} e^{-\frac{1}{2}y} y^{\ell} L_{n+\ell}^{2\ell+1}(y) \,.$$

The associated Laguerre polynomials are defined as

$$L_p^q(y) = \frac{d^q}{dy^q} L_p(y) \quad \text{with} \quad L_p(y) = e^y \frac{d^p}{dy^p} \left(y^p e^{-y} \right) \,.$$

For example,

$$\begin{aligned}
L_1^1(y) &= \frac{d}{dy} L_1(y) = -1 \,, \quad n = 1 \,, \ell = 0 \\
L_2^1(y) &= \frac{d}{dy} L_2(y) = 2y - 4 \,, \quad n = 2 \,, \ell = 0 \\
L_3^3(y) &= \frac{d^3}{dy^3} L_3(y) = -6 \,, \quad n = 2 \,, \ell = 1 \,.
\end{aligned}$$

Generating function for the Laguerre polynomials

The generating function for the Laguerre polynomials is given by

$$G(y,t) = \frac{e^{-yt/(1-t)}}{1-t} = \sum_{n=0}^{\infty} \frac{L_n(y)}{n!} t^n \quad \text{with} \quad t < 1\,.$$

We often apply generating functions

- to derive the special functions, such as
  - (i) Hermite polynomials for harmonic oscillator,
  - (ii) Legendre polynomials for angular momentum eigenfunctions,
  - (iii) Laguerre polynomials for Hydrogen atom,
- to derive the differential equations for special functions, and
- to find the orthonormal relations.

To see that this actually generates Laguerre polynomials we note that

$$\frac{\partial G}{\partial y} = -\frac{t}{1-t} G \quad \text{or} \quad (1-t)\frac{\partial G}{\partial y} = -tG\,.$$

Let us consider each side separately:

$$\begin{aligned}
\text{LHS} &= (1-t)\frac{\partial G}{\partial y} = (1-t)\sum_{n=0}^{\infty}\frac{L_n'}{n!}t^n \\
&= \sum_{n=0}^{\infty}\frac{L_n'}{n!}t^n - \sum_{n=0}^{\infty}\frac{L_n'}{n!}t^{n+1} \\
&= \sum_{n=0}^{\infty}\frac{L_n'}{n!}t^n - \sum_{n=1}^{\infty}\frac{L_{n-1}'}{n-1!}t^n = \sum_{n=0}^{\infty}\frac{L_n'}{n!}t^n - \sum_{n=0}^{\infty}\frac{nL_{n-1}'}{n!}t^n, \quad \text{and} \\
\text{RHS} &= -tG = -t\sum_{n=0}^{\infty}\frac{L_n}{n!}t^n = -\sum_{n=0}^{\infty}\frac{L_n}{n!}t^{n+1} \\
&= -\sum_{n=1}^{\infty}\frac{L_{n-1}}{(n-1)!}t^n \\
&= -\sum_{n=0}^{\infty}\frac{nL_n}{n!}t^n .
\end{aligned}$$

That leads to

$$\sum_{n=0}^{\infty} \frac{L'_n}{n!} t^n - \sum_{n=0}^{\infty} \frac{nL'_{n-1}}{n!} t^n = - \sum_{n=0}^{\infty} \frac{nL_n}{n!} t^n$$

or

$$\begin{aligned} L'_n - nL'_{n-1} &= -nL_{n-1} \quad \text{(I)}, \\ L'_{n+1} - (n+1)L'_n &= -(n+1)L_n \quad \text{(II)} \;. \end{aligned}$$

Furthermore,

$$\begin{aligned} \frac{\partial G}{\partial t} &= -y \left[\frac{1}{1-t} + \frac{t}{(1-t)^2} \right] G + \frac{1}{1-t} G \\ &= \left[-\frac{y}{(1-t)^2} + \frac{1}{(1-t)} \right] G \\ &= \left[\frac{(1-y-t)}{(1-t)^2} \right] G \,. \end{aligned}$$

That leads to

$$(1 - 2t + t^2)\frac{\partial G}{\partial t} = (1 - y - t)G\,.$$

Let us consider each side separately

$$\begin{aligned}
\text{LHS} \quad &= \quad (1 - 2t + t^2)\sum_{n=1}^{\infty}\frac{L_n}{(n-1)!}t^{n-1} \\
&= \quad \sum_{n=1}^{\infty}\left[\frac{L_n}{(n-1)!}t^{n-1} - \frac{2L_n}{(n-1)!}t^n + \frac{L_n}{(n-1)!}t^{n+1}\right] \\
&= \quad \sum_{n=0}^{\infty}\left[\frac{L_{n+1}}{n!}t^n - \frac{2nL_n}{n!}t^n + \frac{nL_n}{n!}t^{n+1}\right] \quad \text{and} \\
\text{RHS} \quad &= \quad (1 - y - t)G \;=\; (1 - y - t)\sum_{n=0}^{\infty}\frac{L_n}{n!}t^n \\
&= \quad (1 - y)\sum_{n=0}^{\infty}\frac{L_n}{n!}t^n - \sum_{n=0}^{\infty}\frac{L_n}{n!}t^{n+1}\,.
\end{aligned}$$

That leads to

$$\sum_{n=0}^{\infty}\left[\frac{L_{n+1}}{n!}t^n - \frac{2nL_n}{n!}t^n + \frac{nL_n}{n!}t^{n+1}\right] = (1-y)\sum_{n=0}^{\infty}\frac{L_n}{n!}t^n - \sum_{n=0}^{\infty}\frac{L_n}{n!}t^{n+1}$$

or

$$\sum_{n=0}^{\infty}[L_{n+1} - 2nL_n - (1-y)L_n]\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} -\frac{(n+1)L_n}{n!}t^{n+1} = -\sum_{n=1}^{\infty}\frac{nL_{n-1}}{(n-1)!}t^n = -\sum_{n=0}^{\infty}\frac{n^2L_{n-1}}{n!}t^n\,.$$

Comparing coefficients, we have

$$L_{n+1} = (2n+1-y)L_n - n^2L_{n-1} \quad \text{(III)}$$

Differentiating (III) with respect to y we obtain

$$\begin{aligned} L'_{n+1} &= (2n+1-y)L'_n - L_n - n^2L'_{n-1} \quad \text{or} \\ n^2L'_{n-1} &= (2n+1-y)L'_n - L'_{n+1} - L_n\,. \quad \text{(IV)} \end{aligned}$$

Multiplying (I) throughout by n and eliminating $n^2 L'_{n-1}$, we have

$$nL'_n + L'_{n+1} - (2n+1-y)L'_n + L_n = -n^2 L'_{n-1}$$

or

$$L'_{n+1} - (n+1-y)L'_n + L_n = -n^2 L'_{n-1}$$

Applying equation (II), we obtain

$$-(n+1)L_n + yL'_n + L_n = -n^2 L_{n-1}$$

or

$$yL'_n - nL_n = -n^2 L_{n-1} \ .$$

Differentiating this with respect to y, we have

$$\begin{aligned} yL''_n + L'_n - nL'_n &= -n^2 L'_{n-1} \text{ or} \\ yL''_n + (1-n)L'_n &= -n^2 L'_{n-1} \end{aligned}$$

We can eliminate $n^2 L'_{n-1}$ by (IV)

$$yL''_n + (1-n)L'_n + (2n+1-y)L'_n - L'_{n+1} - L_n = 0$$

Eliminating L'_{n+1} by (II), we obtain

$$yL''_n + (1-n)L'_n + (n-y)L'_n + (n+1)L_n - L_n = 0$$

or

$$y\frac{d^2L_n}{dy^2} + (1-y)\frac{dL_n}{dy} + nL_n = 0\,.$$

Thus we see that the L_n's satisfy the Laguerre equations.

Generating function for the Associated Laguerre polynomials

The generating function for the associated Laguerre polynomials is

$$G_p(y,t) = \frac{d^p}{dy^p}G(y,t) = (-1)^p\frac{t^p e^{-yt/(1-t)}}{(1-t)^{p+1}} = \sum_{n=p}^{\infty}\frac{L_n^p(y)}{n!}t^n\,.$$

Thus the radial wave function for the Hydrogen atom is

$$R_{n,\ell} = N_{n,\ell} e^{-\frac{1}{2}y} y^{\ell} L_{n+\ell}^{2\ell+1}(y)$$

where

$$\begin{aligned} y &= 2\left(\frac{2\mu|E_n|}{\hbar^2}\right)^{1/2} r = \left(\frac{2\mu e^2}{n\hbar^2}\right) r \quad \text{and} \\ E_n &= -\frac{\mu e^4}{2n^2\hbar^2} . \end{aligned}$$

And the total wave function is

$$\psi_{n,\ell,m}(r,\theta,\phi) = R_{n,\ell}(r) Y_{\ell,m}(\theta,\phi) .$$

The normalization constant $N_{n,\ell}$ can be obtained from the generating function for the associated Laguerre polynomials.

Recall that the total wave function is

$$\psi_{n,\ell,m}(r,\theta,\phi) = R_{n,\ell}(r) Y_{\ell,m}(\theta,\phi) \; .$$

There are orthonormal relations for (a) the total wave function (ψ), (b) the spherical harmonics, and (c) the radial wave function.

The orthonormal relations of the spherical harmonics ($Y_{\ell,m}$) demand that a nonzero contribution comes only if $\ell' = \ell$ and $m' = m$

$$\int Y^*_{\ell',m'}(\theta,\phi) Y_{\ell,m}(\theta,\phi)\, d\Omega = \delta_{\ell'\ell}\delta_{m'm} \quad \text{with} \quad d\Omega = \sin\theta\, d\theta\, d\phi$$

In addition, we have orthonormal relations for the total wave function

$$\int \psi^*_{n',\ell',m'} \psi_{n,\ell,m}\, d^3r = \int \psi^*_{n',\ell',m'} \psi_{n,\ell,m}\, r^2\, dr\, d\Omega = \delta_{n'n}\delta_{\ell'\ell}\delta_{m'm}$$

and the radial wave function

$$\int R^*_{n'\ell'}(r) R_{n\ell}(r)\, r^2\, dr = \delta_{n'n}\delta_{\ell'\ell} \, .$$

Let us look at

$$\int R^*_{n,\ell} R_{n,\ell}\, r^2\, dr = \left(\frac{n\hbar^2}{2\mu e^2}\right)^3 |N_{n,\ell}|^2 \int y^2 e^{-y} y^{2\ell} L^{2\ell+1}_{n+\ell}(y) L^{2\ell+1}_{n+\ell}(y) dy$$

$$= \left(\frac{n\hbar^2}{2\mu e^2}\right)^3 |N_{n,\ell}|^2 \int y^{2\ell+2} e^{-y} L^{2\ell+1}_{n+\ell}(y) L^{2\ell+1}_{n+\ell}(y) dy\,.$$

We now write the associated Laguerre polynomials in terms of their generating functions

$$L^{2\ell+1}_{n+\ell}(y) = \frac{\partial^{n+\ell}}{\partial t^{n+\ell}} \left[(-1)^{2\ell+1} \frac{t^{2\ell+1}}{(1-t)^{2\ell+2}} e^{-yt/(1-t)}\right]_{t=0}.$$

The normalization constant of the radial wave function can be determined with

$$\begin{aligned}
&\int y^{2\ell+2} e^{-y} L_{n+\ell}^{2\ell+1}(y) L_{n+\ell}^{2\ell+1}(y) dy \\
= \quad & \frac{\partial^{n+\ell}}{\partial t^{n+\ell}} \frac{\partial^{n+\ell}}{\partial x^{n+\ell}} \left[\frac{(tx)^{2\ell+1}}{(1-t)^{2\ell+2}(1-x)^{2\ell+2}} \right. \\
& \quad \left. \times \int y^{2\ell+2} e^{-y} e^{-yt/(1-t)} e^{-yx/(1-x)} dy \right]_{t=x=0} \\
= \quad & \frac{\partial^{n+\ell}}{\partial t^{n+\ell}} \frac{\partial^{n+\ell}}{\partial x^{n+\ell}} \left[\frac{(tx)^{2\ell+1}}{(1-t)^{2\ell+2}(1-x)^{2\ell+2}} \right. \\
& \quad \left. \times \int y^{2\ell+2} e^{-y(1-tx)/((1-t)(1-x))} \, dy \right]_{t=x=0} .
\end{aligned}$$

Changing

$$y \to \left[\frac{(1-xt)}{(1-t)(1-x)}\right]^{-1} z$$

we obtain

$$\begin{aligned}
& \frac{\partial^{n+\ell}}{\partial t^{n+\ell}} \frac{\partial^{n+\ell}}{\partial x^{n+\ell}} \left[\frac{(tx)^{2\ell+1}}{(1-t)^{2\ell+2}(1-x)^{2\ell+2}}\right. \\
& \qquad \left. \times \frac{(1-t)^{2\ell+3}(1-x)^{2\ell+3}}{(1-xt)^{2\ell+3}} \int z^{2\ell+2} e^{-z}\, dz\right]_{t=x=0} \\
= \quad & \frac{\partial^{n+\ell}}{\partial t^{n+\ell}} \frac{\partial^{n+\ell}}{\partial x^{n+\ell}} \left[\frac{(tx)^{2\ell+1}(1-t)(1-x)}{(1-xt)^{2\ell+3}} \Gamma(2\ell+3)\right]_{t=x=0} \\
= \quad & (2n)\frac{[(n+\ell)!]^3}{(n-\ell-1)!}
\end{aligned}$$

The normalization becomes

$$\begin{aligned}\int R^*_{n,\ell} R_{n,\ell}\, r^2\, dr &= \left(\frac{n\hbar^2}{2\mu e^2}\right)^3 |N_{n,\ell}|^2 \int y^2 e^{-y} y^{2\ell} L^{2\ell+1}_{n+\ell}(y) L^{2\ell+1}_{n+\ell}(y) dy \\ &= \left(\frac{n\hbar^2}{2\mu e^2}\right)^3 |N_{n,\ell}|^2 \int y^{2\ell+2} e^{-y} L^{2\ell+1}_{n+\ell}(y) L^{2\ell+1}_{n+\ell}(y) dy \\ &= 1\,,\end{aligned}$$

or

$$\left(\frac{n\hbar^2}{2\mu e^2}\right)^3 |N_{n,\ell}|^2 (2n) \frac{[(n+\ell)!]^3}{(n-\ell-1)!} = 1\,.$$

Choosing $N_{\ell,m}$ to be real and negative, we obtain

$$N_{\ell,m} = N^*_{\ell,m} = -\left[\left(\frac{2\mu e^2}{n\hbar^2}\right)^3 \frac{(n-\ell-1)!}{(2n)[(n+\ell)!]^3}\right]^{1/2}\,.$$

Thus the normalized radial wave functions are

$$R_{n,\ell} = -\left[\left(\frac{2m_e e^2}{n\hbar^2}\right)^3 \frac{(n-\ell-1)!}{(2n)[(n+\ell)!]^3}\right]^{1/2} \times e^{-\frac{1}{2}y} y^\ell L_{n+\ell}^{2\ell+1}(y) \quad \text{with}$$

$$y = 2\left(\frac{2m_e|E|}{\hbar^2}\right)^{1/2} r = \left(\frac{2m_e e^2}{n^2\hbar^2}\right) r \quad \text{and} \quad E_n = -\frac{m_e e^4}{2n^2\hbar^2}\,.$$

The first three radial functions are

$$R_{10} = 2\left(\frac{m_e e^2}{\hbar^2}\right)^{3/2} e^{-(m_e e^2/\hbar^2)r} = 2\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}\,,$$

$$R_{20} = \left(\frac{1}{2a_0}\right)^{3/2}\left(2-\frac{r}{a_0}\right) e^{-r/(2a_0)}\,,$$

$$R_{21} = \left(\frac{1}{2a_0}\right)^{3/2}\left(\frac{r}{\sqrt{3}a_0}\right) e^{-r/(2a_0)} \quad \text{with}$$

$$a_0 = \frac{\hbar^2}{m_e e^2} \quad \text{(Bohr radius).}$$

Radial wave function of the Hydrogen Atom

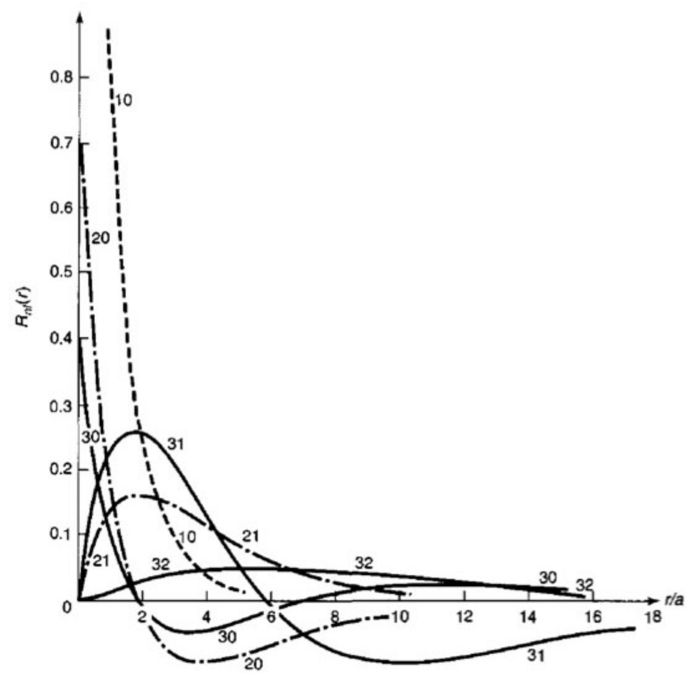

Figure 4.4: Graphs of the first few hydrogen radial wave functions, $R_{nl}(r)$.

Figure 1: Hydrogen radial wave function.

6.3 Fundamental Quantities Associated with Hydrogen Atom

Looking at the wave functions for the Hydrogen atom, we notice that there is a fundamental length scale that enters the solutions:

$$a_0 = \frac{\hbar^2}{m_e e^2}$$

that is the Bohr radius.

In terms of this quantity, we can write down the radial solutions as

$$R_{n,\ell}(r) \sim e^{-r/na_0} \left(\frac{2r}{na_0}\right)^{\ell} L_{n+\ell}^{2\ell+1}\left(\frac{2r}{na_0}\right)$$

Remembering that $L_{n+\ell}^{2\ell+1}$ is a polynomial of order $n-\ell-1$, the most dominant behavior for large r ($r \gg a_0$) would be

$$R_{n,\ell}(r) \sim (r)^{n-1} e^{-r/na_0} \ .$$

Then the probability for finding the electron in a special shell of radius r and thickness dr

$$\int_{\Omega} \psi^* \psi r^2 dr d\Omega \quad \sim \quad R^2_{n,\ell}(r) r^2 dr$$
$$= \quad (r)^{2n} e^{-2r/na_0} dr \; .$$

We can thus determine the radius of maximum probability as

$$\frac{d}{dr}\left((r)^{2n} e^{-2r/na_0}\right) = 2n r^{2n-1} e^{-2r/na_0} - \frac{2}{na_0}(r)^{2n} e^{-2r/na_0}$$
$$= \left(\frac{1}{na_0}\right)(r - n^2 a_0)(r)^{2n-1} e^{-2r/na_0}$$
$$= 0$$

or

$$r_* = n^2 a_0 \; .$$

Thus we see that the Bohr radius a_0 is the most probable value of r in the ground state and thus defines the natural size of the Hydrogen atom. We also see that $\langle r \rangle$ grows as n^2.

This theory also possesses a natural energy scale. Thus we define

$$R_y = \frac{m_e e^4}{2\hbar^2} = \text{ Rydberg.}$$

The the energy levels for the Hydrogen atom become

$$E_n = -\frac{R_y}{n^2} \ .$$