PHYS 3803: Quantum Mechanics I, Spring 2021

Lecture 24, April 27, 2021 (Tuesday)

- Reading: Hydrogen Atom—Griffiths 4.2
- Assignments: Problem Set 10 due April 30 (Friday).
  Submit your homework assignments to Canvas.

Topics for Today: Hydrogen Atom [Griffiths 4.2]

5.3 Schrödinger equation for spherically symmetric potentials

6.1 Relative Motion of Two Particles

6.2 Introduction to the Hydrogen Atom

Next Lecture: Hydrogen Atom [Griffiths 4.2]

6.3 Fundamental Quantities Associated with Hydrogen Atom

6.4 Numerical Estimates

5.3 Schrödinger equation for spherically symmetric potentials

We can write the Laplacian as

$$\begin{aligned}\nabla^2 &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{L^2}{\hbar^2 r^2}\,.\end{aligned}$$

And the Schrödinger equation for this system becomes

$$H\psi(r,\theta,\phi)=E\psi(r,\theta,\phi)\quad\text{(eigenvalue equation)}\quad\text{and}$$

$$\begin{aligned}\left[-\frac{\hbar^2}{2m}\nabla^2+V(r)\right]\psi(r,\theta,\phi) &= E\psi(r,\theta,\phi)\\ \left\{-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{L^2}{\hbar^2 r^2}\right]+V(r)\right\}\psi(r,\theta,\phi) &= E\psi(r,\theta,\phi)\,.\end{aligned}$$

Let us now consider a separable solution

$$\psi(r,\theta,\phi) = R(r)F(\theta,\phi) = R_{E\ell}(r)P_{\ell m}(\theta)\Phi_m(\phi)\,.$$

Thus we have

$$\begin{aligned} \frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2mr^2}{\hbar^2}[E - V(r)]R(r) &= \lambda R(r) \quad \text{and} \\ L^2 F(\theta,\phi) &= \lambda\hbar^2 F(\theta,\phi)\,, \end{aligned}$$

where $\lambda\hbar^2 = \ell(\ell+1)\hbar^2$ is the eigenvalue of the operator L^2.

The $\phi-$equation now becomes

$$\frac{L_z^2}{(i\hbar)^2}\Phi = \frac{d^2\Phi}{d\phi^2} = -\alpha\Phi(\phi) = -m^2\Phi(\phi)$$

and the normalized eigenfunction in the ϕ basis is

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}\,,\;\; 0 \le \phi \le 2\pi\,, \quad -\ell \le m \le \ell$$

where $m\hbar$ is the eigenvalue of L_z with the eigenfunction $\Phi_m(\phi)$.

The $\theta-$equation now becomes

$$\begin{aligned}\sin\theta\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + \left[\ell(\ell+1)\sin^2\theta - m^2\right]\Theta &= 0 \quad \text{or} \\ (1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + [\ell(\ell+1) - \frac{m^2}{1-x^2}]\Theta_{\ell,m} &= 0\,.\end{aligned}$$

The solution now depends on two quantum numbers ℓ and m with

$$z(x) = z_{\ell,m}(x) = \sum a_k x^k \quad \text{and} \quad x = \cos\theta\,.$$

This is a polynomial of order $k = \ell - |m|$, and the $\theta-$solution becomes

$$\Theta_{\ell,m} = (1-x^2)^{\frac{|m|}{2}} z_{\ell,m}(x) = P_{\ell,m}(x)\,.$$

For $m = 0$, the equation is known as the Legendre equation and the solutions of the equation are known as the Legendre functions $P_\ell(x)$:

$$(1-x^2)\frac{d^2P_\ell}{dx^2} - 2x\frac{dP_\ell}{dx} + \ell(\ell+1)P_\ell(x) = 0\,.$$

The $\Theta_{\ell,m}-$functions are related to the Legendre functions by

$$\Theta_{\ell,m}(x) = (1-x^2)^{\frac{|m|}{2}} \frac{d^{|m|} P_\ell(x)}{dx^{|m|}} = P_{\ell,m}(x)$$

for $\ell \geq |m|$, and are known as the associated Legendre functions.

The complete angular part of the solution is

$$F_{\ell,m}(\theta,\phi) = Y_{\ell,m}(\theta,\phi) \equiv \langle \theta,\phi|\ell,m\rangle = \frac{N_{\ell,m}}{\sqrt{2\pi}} P_{\ell,m}(\theta) e^{im\phi}$$

where

- $Y_{\ell,m}$'s are called the spherical Harmonics, and
- $N_{\ell,m}$'s are normalization constants.

The normalization constant is determined to be

$$N_{\ell,m} = N^*_{\ell,m} = \pm\sqrt{\frac{2\ell+1}{2}\frac{(\ell-|m|)!}{(\ell+|m|)!}}\,.$$

Conventionally we choose the sign to be $(-1)^m$ for $m > 0$ and $+$ for $m \leq 0$. Therefore, the normalized angular solutions are

$$Y_{\ell,m} = \epsilon\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}P_{\ell,m}(\cos\theta)e^{im\phi}$$

where $\epsilon = (-1)^m$ for $m > 0$ and $\epsilon = +1$ for $m \leq 0$.

The complete solution to the Schrödinger equation is

$$\psi_{E,\ell,m}(r,\theta,\phi) = R(r)Y_{\ell,m}(\theta,\phi)\,.$$

The radial part $R(r)$ is determined by the dynamics of the system.

The equation of the radial part is

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{2\mu r^2}{\hbar^2}[E-V(r)]R(r)=\lambda R(r)=\ell(\ell+1)R(r)$$

or

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{2\mu r^2}{\hbar^2}\left[E-V(r)-\frac{\hbar^2\ell(\ell+1)}{2\mu r^2}\right]R(r)=0\,.$$

N.B. From now on the mass of a particle will be denoted by μ.

There are several things to note.

- A nonzero angular momentum implies the presence of an additional potential.
- Furthermore, if we differentiate and calculate the force, we notice it pushes the particle away from the center of the coordinate system and lies along the radial direction.
- Thus a nonzero angular momentum gives rise to a centrifugal barrier that is very strong at short distances.

Parity

We can now study the question of parity in the 3-dimensional case. Parity means reflecting a vector through the origin. Thus in spherical coordinates a vector under the parity transformation is denoted by

$$\Pi(r,\theta,\phi) \rightarrow (r,\pi-\theta,\pi+\phi)$$

where $\Pi =$ the parity operator. Since the radial part of the vector does not change under reflection, only the angular part of the solution would be affected.

$$\Pi Y_{\ell,m}(\theta,\phi) \rightarrow Y_{\ell,m}(\pi-\theta,\pi+\phi)$$

Now we have

$$\begin{aligned} \Pi e^{im\phi} \quad &\rightarrow \quad e^{im(\pi+\phi)} = (-1)^{|m|} e^{im\phi} \\ \Pi \cos\theta \quad &\rightarrow \quad \cos(\pi-\theta) = -\cos\theta = -x \end{aligned}$$

Thus

$$
\begin{aligned}
P_{\ell,m}(x) &= (1-x^2)^{\frac{|m|}{2}} \frac{d^{|m|}}{dx^{|m|}} P_\ell(x) \\
P_\ell(x) &= \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2-1)^\ell \\
\Pi P_{\ell,m}(x) &\propto (-1)^{\ell+|m|} (1-x^2)^{\frac{|m|}{2}} \frac{d^{\ell+|m|}}{dx^{\ell+|m|}} (x^2-1)^\ell \\
&\propto (-1)^{\ell+|m|} P_{\ell,m}
\end{aligned}
$$

Then we have

$$
\Pi Y_{\ell,m} = (-1)^{|m|}(-1)^{\ell+|m|} Y_{\ell,m}(\theta,\phi) = (-1)^\ell Y_{\ell,m}(\theta,\phi).
$$

- The angular eigenfunctions are definite parity eigenstates.
- Their parity is completely determined by the orbital angular momentum or the $\ell-$quantum number.
- All the $(2\ell+1)$ states with different $m-$quantum numbers have the same parity given by $(-1)^\ell$.

Examples of Spherically Symmetric Potentials:

- The 3-dimensional isotropic oscillator

$$H = \frac{P^2}{2\mu} + \frac{1}{2}\mu\omega^2\, r^2\,, \quad r = |\vec{r}|\,.$$

- The square well potential

$$V(r) = \begin{cases} 0 & \text{for } r > a, \text{ and} \\ -V_0, & \text{for } r \le a. \end{cases}$$

for $V_0 > 0$ and $|E| < V_0$.

- The Hydrogen atom

$$H = \frac{P^2}{2\mu} - \frac{e^2}{r}$$

where $e =$ the charge of a proton.

6.1 Relative Motion of Two Particles

Let us consider an isolated system of two interacting particles of masses m_1 and m_2 at positions $\vec{r}_1$ and $\vec{r}_2$, and they interact through a potential that depends on the relative separation between the two particles.

The motion of the system can be split into two parts: a part that describes the motion of the center of mass and another which describes the relative motion of the two particles.

Classical System

The Lagrangian is

$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - V(\vec{r}_1 - \vec{r}_2) \; .$$

We often define the coordinates

$$\vec{r} \equiv \vec{r}_1 - \vec{r}_2 \quad \text{and} \quad \vec{R} \equiv \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{(m_1 + m_2)}$$

where R is the center of mass of the system, and then

$$\vec{r}_1 = \vec{R} + \frac{m_2}{m_1 + m_2}\vec{r} \quad \text{and} \quad \vec{r}_2 = \vec{R} - \frac{m_1}{m_1 + m_2}\vec{r}\,.$$

Thus the Lagrangian becomes

$$\begin{aligned}
L &= \frac{1}{2}m_1\left(\dot{\vec{R}} + \frac{m_2}{m_1+m_2}\dot{\vec{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\vec{R}} - \frac{m_1}{m_1+m_2}\dot{\vec{r}}\right)^2 - V(\vec{r}) \\
&= \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}\left(\frac{m_1 m_2^2}{(m_1+m_2)^2} + \frac{m_1^2 m_2}{(m_1+m_2)^2}\right)\dot{\vec{r}}^2 - V(\vec{r}) \\
&= \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}\left(\frac{m_1 m_2}{(m_1+m_2)}\right)\dot{\vec{r}}^2 - V(\vec{r}) \\
&= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - V(\vec{r})\,.
\end{aligned}$$

where

- $M = m_1 + m_2 =$ the total mass, and
- $\mu = m_1 m_2/(m_1 + m_2) =$ the reduced mass of the system.

The conjugate momenta corresponding to these new coordinates are

$$\begin{aligned}\vec{Q} &= \frac{\partial L}{\partial \vec{R}} = M\dot{\vec{R}} = M\frac{(m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2)}{M} \\ &= m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2 = \vec{p}_1 + \vec{p}_2\end{aligned}$$

which is the sum of the individual momentum, and

$$\begin{aligned}\vec{p} &= \frac{\partial L}{\partial \vec{r}} = \mu\dot{\vec{r}} = \frac{m_1 m_2}{m_1 + m_2}(\dot{\vec{r}}_1 - \dot{\vec{r}}_2) \\ &= \frac{m_2\vec{p}_1 - m_1\vec{p}_2}{m_1 + m_2}\end{aligned}$$

Thus the Hamiltonian can be written as

$$H = \frac{\vec{Q}^2}{2M} + \frac{\vec{p}^2}{2\mu} + V(\vec{r})\ .$$

The motion of the system can be described by the motion of two fictitious particles–one with the total mass and the coordinates of the center of mass of the system and the other with a reduced mass and the relative coordinates. Furthermore, since the variable $\vec{Q}$ is cyclic

$$\dot{\vec{Q}} = 0 \; .$$

The total momentum of the system is constant and we can go to the frame of the center of mass in which case

$$\vec{Q} = 0$$

and the Hamiltonian becomes

$$H = \frac{\vec{p}^2}{2\mu} + V(\vec{r}) \, .$$

Thus we see that the problem of two interacting particles reduces in the center of mass frame to that of a single particle with a reduced mass and with the relative coordinates.

Quantum System

The Hamiltonian is

$$\begin{aligned} H &= \frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_2} + V(\vec{r}_1 - \vec{r}_2) \\ &= \frac{\vec{Q}^2}{2M} + \frac{\vec{p}^{\,2}}{2\mu} + V(\vec{r}) \,. \end{aligned}$$

Thus we see that the initial Hamiltonian can be equivalently written as a sum of two uncoupled terms. Quantum mechanically, we know that the operators would obey the following commutation relations:

$$\begin{aligned} [r_{1i}, p_{1j}] &= i\hbar\delta_{ij} \\ [r_{2i}, p_{2j}] &= i\hbar\delta_{ij} \\ [r_{1i}, r_{1j}] &= 0\,, \quad [r_{2i}, r_{2j}] \;=\; 0\,, \\ [p_{1i}, p_{1j}] &= 0\,, \quad [p_{2i}, p_{2j}] \;=\; 0\,, \\ [r_{1i}, p_{2j}] &= 0\,, \quad [r_{2i}, p_{1j}] \;=\; 0\,, \\ [r_{1i}, r_{2j}] &= 0\,, \quad [p_{1i}, p_{2j}] \;=\; 0\,. \end{aligned}$$

From this we can show that

$$[R_i, R_j] = 0 = [r_i, r_j] \ , \quad [Q_i, Q_j] = 0 = [q_i, q_j] \ ,$$

and

$$[R_i, r_j] = 0 = [Q_i, q_j] \ , [R_i, q_j] = 0 = [r_i, Q_j] \ .$$

However, we have

$$\begin{aligned} [R_i, Q_j] &= i\hbar\delta_{ij} \\ [r_i, q_j] &= i\hbar\delta_{ij} \ . \end{aligned}$$

Thus $(\vec{r}, \vec{p})$ and $(\vec{R}, \vec{Q})$ behave like two pairs of conjugate variables. Thus in the coordinate basis, we can write

$$\begin{aligned} \vec{p} &= -i\hbar\vec{\nabla} = -i\hbar\frac{\partial}{\partial\vec{r}} \\ \vec{Q} &= -i\hbar\vec{\nabla}_M = -i\hbar\frac{\partial}{\partial\vec{R}} \ . \end{aligned}$$

Since the two sets $(\vec{r}, \vec{p})$ and $(\vec{R}, \vec{Q})$ commute with each other, the Hamiltonian can be written as the direct sum of two Hamiltonians:

$$H = H_M + H_r$$

where H_M is the Hamiltonian associated with the motion of the center of mass and H_r is associated with the relative motion of the two particles. And since

$$[H_M, H_r] = 0$$

they can be simultaneously diagonalized. The Hilbert space of the system becomes a product space of two Hilbert spaces

$$\mathcal{E} = \mathcal{E}_M \otimes \mathcal{E}_r$$

where $(H_M, \vec{R}, \vec{Q})$ act only on $\mathcal{E}_M$ and $(H_r, \vec{r}, \vec{p})$ act only on $\mathcal{E}_r$. A general state of $\mathcal{E}$ can be written as

$$|\alpha_M, \beta_r\rangle = |\alpha_M\rangle \otimes |\beta_r\rangle \, .$$

The situation here is exactly similar to the higher dimensional oscillator in Cartesian coordinates. Thus a general wave function becomes

$$\langle \vec{r}, \vec{R} | \psi_M, \psi_r \rangle = \psi_M(\vec{R}) \psi_r(\vec{r}) \, .$$

The Schödinger equation now splits into two equations

$$-\frac{\hbar^2}{2M} \vec{\nabla}_M^2 \psi_M(\vec{R}) = E_M \psi_M(\vec{R}) \quad \text{and} \quad \left[-\frac{\hbar^2}{2\mu} \vec{\nabla}^2 + V(\vec{r}) \right] \psi(\vec{r}) = E_r \psi_r(\vec{r})$$

where $E_r + E_M = E =$ the total energy of the system.

The first equation is the equation of motion for a free particle and it is easy to solve

$$\psi_M(\vec{Q}) = e^{\frac{i}{\hbar} \vec{Q} \cdot \vec{R}} \quad \text{with} \quad E_M = \frac{Q^2}{2M} \, .$$

The interesting dynamics is in the other equation for relative motion

$$\left[-\frac{\hbar^2}{2\mu} \vec{\nabla}^2 + V(\vec{r}) \right] \psi(\vec{r}) = E_r \psi_r(\vec{r}) \, .$$

6.2 Hydrogen Atom

Here we are consider the motion of an electron in the electromagnetic potential of a proton inside a nucleus. Thus

$$m_1 = m_p = 938\ \mathrm{MeV} \quad \text{and} \quad m_2 = m_e = 0.511\ \mathrm{MeV}\ .$$

The Coulomb potential of the proton is e/r and, therefore, the potential energy of the system is

$$V(r) = -\frac{e^2}{r}\ .$$

The Hamiltonian associated with the relative motion is

$$H = -\frac{\hbar^2}{2\mu}\nabla^2 - \frac{e^2}{r}\ .$$

We can write the solution as

$$\psi_{E,\ell,m}(r,\theta,\phi) = R_{E,\ell}(r) Y_{\ell,m}(\theta,\phi)\ .$$

Defining $R_{E,\ell} \equiv u_{E,\ell}(r)/r$, we can write the equation for $u_{E,\ell}$ as

$$\frac{d^2 u_{E,\ell}}{dr^2} + \frac{2\mu}{\hbar^2}\left(E + \frac{e^2}{r} - \hbar^2 \frac{\ell(\ell+1)}{2\mu r^2}\right) u_{E,\ell} = 0 \ .$$

We are interested in the bound state solutions, namely, $E < 0$. For $E = -|E|$, the equation becomes asymptotically

$$r \to \infty, \quad \frac{d^2 u_{E,\ell}}{dr^2} + \frac{2\mu E}{\hbar^2} u_{E,\ell} \quad = \quad 0 \ , \quad \text{with}$$

$$\lim_{r\to\infty} \ u_{E,\ell} \to e^{-\left(\frac{2\mu|E|}{\hbar^2}\right)^{1/2} r} \quad = \quad e^{-\frac{1}{2}y}$$

where we have defined the dimensionless parameter

$$y = 2\left(\frac{2\mu|E|}{\hbar^2}\right)^{1/2} r \quad \text{and} \quad \frac{d}{dr} = 2\left(\frac{2\mu|E|}{\hbar^2}\right)^{1/2} \frac{d}{dy} \ .$$

In terms of the y variables, the equation now becomes

$$\frac{d^2 u_{E,\ell}}{dy^2} + \frac{1}{4}\left(-1 + \frac{e^2}{r|E|} - \frac{\ell(\ell+1)\hbar^2}{2\mu|E|r^2}\right) u_{E,\ell} = 0\,,$$

or

$$\frac{d^2 u_{E,\ell}}{dy^2} + \left(-\frac{1}{4} + \frac{\lambda}{y} - \frac{\ell(\ell+1)}{y^2}\right) u_{E,\ell} = 0\,,$$

where

$$\lambda = \left(\frac{\mu}{2\hbar^2|E|}\right)^{1/2} e^2 = \left(\frac{\mu e^4}{2\hbar^2|E|}\right)^{1/2} \;.$$

Near the origin ($r \to 0$ and $y \to 0$) the equation becomes

$$r \to 0, \quad \frac{d^2 u_{E,\ell}}{dy^2} - \frac{\ell(\ell+1)}{y^2} u_{E,\ell} = 0\;.$$

and

$$\lim_{r\to 0}\; u_{E,\ell}(y) \to y^{\ell+1}\;.$$

This suggests a general solution of the form

$$\begin{aligned} u_{E,\ell}(y) &= e^{-\frac{1}{2}y} v(y) \\ &= e^{-\frac{1}{2}y} y^{\ell+1} \sum_{k=0}^{\infty} a_k y^k = e^{-\frac{1}{2}y} \sum_{k=0}^{\infty} a_k y^{k+\ell+1} \end{aligned}$$

and

$$\frac{d^2 u_{E,\ell}}{dy^2} = e^{-\frac{1}{2}y} \left[\frac{1}{4} v - v' + v'' \right] .$$

Thus the equation for v becomes

$$\begin{aligned} e^{\frac{1}{2}y} \left[v'' - v' + \frac{1}{4} v + \left(-\frac{1}{4} + \frac{\lambda}{y} - \frac{\ell(\ell+1)}{y^2} \right) v \right] &= 0 \quad \text{or} \\ \frac{d^2 v}{dy^2} - \frac{dv}{dy} + \left[\frac{\lambda}{y} - \frac{\ell(\ell+1)}{y^2} \right] v &= 0 . \end{aligned}$$

Let us now use the power series solution

$$v(y) = \sum_{k=0}^{\infty} a_k y^{k+\ell+1}, \quad \frac{dv}{dy} = \sum_{k=0}^{\infty} (k+\ell+1) a_k y^{k+\ell} \quad \text{and}$$

$$\frac{d^2 v}{dy^2} = \sum_{k=0}^{\infty} (k+\ell+1)(k+\ell) a_k y^{k+\ell-1}$$

Thus the equation becomes

$$\sum_{k=0}^{\infty} \Big[(k+\ell+1)(k+\ell) a_k y^{k+\ell-1} - (k+\ell+1) a_k y^{k+\ell} + \lambda a_k y^{k+\ell} - \ell(\ell+1) a_k y^{k+\ell-1} \Big] = 0, \quad \text{or}$$

$$\sum_{k=0}^{\infty} \Big[\Big((k+\ell+1)(k+\ell) - \ell(\ell+1) \Big) a_k y^{k+\ell-1} - (k+\ell+1-\lambda) a_k y^{k+\ell} \Big] = 0$$

Looking at the lowest power of y, that is $y^{\ell-1}$, we see

$$[\ell(\ell+1) - \ell(\ell+1)]a_0 = 0$$

Then we have $a_0 \neq 0$ or a_0 is arbitrary.

The next coefficient, that is y^ℓ, gives

$$[(\ell+2)(\ell+1) - \ell(\ell+1)]a_1 - (\ell+1-\lambda)a_0 = 0$$

Then we have $a_1 \neq 0$ if $a_0 \neq 0$.

In general the recursion relation would connect a_{k+1} to a_k. Thus looking at the coefficient of $y^{k+\ell}$ we have

$$[(k+1+\ell+1)(k+1+\ell) - \ell(\ell+1)]a_{k+1} = (k+\ell+1-\lambda)a_k$$

or

$$a_{k+1} = \frac{(k+\ell+1-\lambda)}{[(k+1+\ell+1)(k+1+\ell) - \ell(\ell+1)]}a_k$$

Note that

$$\begin{aligned}(k+\ell+2)(k+\ell+1)-\ell(\ell+1) &= k^2+2\ell k+3k+2\ell+2 \\ &= k^2+2\ell k+2k+k+2\ell+2 \\ &= (k+1)(k+2\ell+2)\end{aligned}$$

Thus for large k

$$\frac{a_{k+1}}{a_k} \rightarrow \frac{1}{k}$$

This is the behavior of the series e^y for large higher orders and thus unless the series terminates this would lead to an unphysical solution. The series terminates if

$$k+\ell+1-\lambda=0$$

or

$$\lambda = \left(\frac{\mu e^4}{2\hbar|E|}\right)^{1/2} = k+\ell+1 = n$$

or

$$
\begin{aligned}
|E| &= \frac{\mu e^4}{2\hbar^2 n^2} \\
E_n &= -|E_n| = -\frac{\mu e^4}{2\hbar^2 n^2} \,.
\end{aligned}
$$

Since both k and ℓ take positive integer values, n also take positive integer values. Even when ℓ and k are both equal to zero, $n = 1$. Thus the allowed values for n are

$$n = 1, 2, 3, \cdots, \infty$$

In addition,

$$\ell = n - k - 1 = n - 1, n - 2, \cdots, 0$$

These are the allowed values of the orbital angular momentum for a given value of n.