PHYS 3803: Quantum Mechanics I, Spring 2021

Lecture 23, April 22, 2021 (Thursday)

- Handout: Solutions to Problem Set 9.
- Reading: Angular Momentum: Griffiths 4.1 and 4.3
- Assignments: Problem Set 10 due April 30 (Friday).
  Submit your homework assignments to Canvas.

Topics for Today: Angular Momentum [Griffiths 4.3]

5.3 Schrödinger equation for spherically symmetric potentials

Next Lecture: Hydrogen Atom [Griffiths 4.2]

6.1 Relative Motion of Two Particles

6.2 Introduction to the Hydrogen Atom

5.2 Rotations and Angular Momentum

The θ equation is

$$\frac{d}{d\theta}\Theta_{\ell,\ell}(\theta) - \ell \cot\theta \Theta_{\ell,\ell}(\theta) = 0\,.$$

That has the solution

$$\Theta_{\ell,\ell}(\theta) = A(\sin\theta)^{\ell}\,, \quad \text{and} \quad U_{\ell,\ell}(r,\theta,\phi) = R_{\ell,\ell}(r)(\sin\theta)^{\ell} e^{i\ell\phi}\,.$$

Note that rotation only affects the angular parts.

- The radial component should not depend on any angular momentum quantum numbers.
- It should be the same for all wave functions of different angular momentum quantum numbers and is determined by the dynamics of the system.

Thus

$$U_{\ell,\ell}(r,\theta,\phi) = R_{\ell,\ell}(r)(\sin\theta)^{\ell} e^{i\ell\phi}\,.$$

Any other wave function can be obtained from this by using the lowering operator. Thus

$$\begin{aligned}
|\ell,\ell-1\rangle &= \frac{1}{[\ell(\ell+1)-\ell(\ell-1)]^{1/2}\hbar} L_-|\ell,\ell\rangle \;=\; \frac{1}{\sqrt{2\ell}\,\hbar} L_-|\ell,\ell\rangle \\
U_{\ell,\ell-1}(r,\theta,\phi) &= \frac{1}{\sqrt{2\ell}\,\hbar}(-1)\hbar e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right)U_{\ell,\ell}(r,\theta,\phi) \\
&= -\frac{2}{\sqrt{2\ell}\,\hbar} e^{-i\phi}\cdot\frac{\partial}{\partial\theta}U_{\ell,\ell}(r,\theta,\phi)
\end{aligned}$$

where we have used

$$\begin{aligned}
\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)U_{\ell,\ell}(r,\theta,\phi) &= 0\,, \quad \text{or} \\
+i\cot\theta\frac{\partial}{\partial\phi}U_{\ell,\ell}(r,\theta,\phi) &= -\frac{\partial}{\partial\theta}U_{\ell,\ell}(r,\theta,\phi)\,.
\end{aligned}$$

Thus

$$\begin{aligned} U_{\ell,\ell-1}(r,\theta,\phi) &= \frac{(-1)}{\sqrt{2\ell}} \cdot 2e^{-i\phi} R(r) e^{i\ell\phi} \frac{d}{d\theta} (\sin\theta)^{\ell} \\ &= \frac{(-1)}{\sqrt{2\ell}} \cdot 2R(r) e^{i(\ell-1)\phi} \cdot \ell (\sin\theta)^{\ell-1} \cos\theta \\ &= -\sqrt{2\ell} R(r) e^{i(\ell-1)\phi} (\sin\theta)^{\ell-1} \cos\theta \,. \end{aligned}$$

A general wave function $U_{\ell,m}(r,\theta,\phi)$ can be obtained by applying the lowering operator $\ell - m$ times with suitable normalization.

The interesting conclusions of this operator method is that the angular momentum operator L^2 has eigenvalues of the form

(i) $L^2|\ell, m\rangle = \ell(\ell+1)\hbar^2|\ell, m\rangle$.

(ii) ℓ takes integer as well as half integral values.

This is certainly a triumph of quantum mechanics.

For the energy associated with rotation, the Hamiltonian is denoted by

$$H = \frac{L^2}{2I}$$

with the eigenvalue

$$H|\ell, m\rangle = E_\theta |\ell, m\rangle$$

where I is the moment of inertia of the system.

In the old quantum theory, someone suggested the eigenvalues of L^2 as

$$L^2|\ell, m\rangle = \ell^2\hbar^2|\ell, m\rangle \quad \text{and} \quad E_\theta = \frac{\ell^2\hbar^2}{2I} \quad \text{with} \quad \ell = 0, 1, 2, \cdots .$$

That leads to the separation of energy levels in molecules in the proportions

$$\Delta E_{10} : \Delta E_{21} : \Delta E_{32} : \Delta E_{43} : \cdots = 1 : 3 : 5 : 7 : \cdots$$

However, this was not observed.

The correct quantum mechanics predicts

$$L^2|\ell, m\rangle = \ell(\ell+1)\hbar^2|\ell, m\rangle \quad \text{with} \quad \ell = 0, 1, 2, \cdots .$$

The observed energy separations were in the proportions

$$\Delta E_{10} : \Delta E_{21} : \Delta E_{32} : \Delta E_{43} : \cdots = 1 : 2 : 3 : 4 : \cdots = 2 : 4 : 6 : 8 : \cdots .$$

That confirms the correct quantum theory with

$$L^2|\ell, m\rangle = \ell(\ell+1)\hbar^2|\ell, m\rangle \quad \text{and} \quad E_\theta = \frac{\ell(\ell+1)\hbar^2}{2I} \quad \text{with} \quad \ell = 0, 1, 2, \cdots$$

- The angular momentum eigenvalues take integral values and the extra term arise from the noncommutativity of different components of the angular momentum operators.
- If we solve Schrödinger equations, we would only obtain integral values for angular momentum eigenvalue.
- The operator method, on the other hand, allows half integral eigenvalues as well.

5.3 Schrödinger equation for spherically symmetric potentials

Spherical symmetry $\leftrightarrow$ Rotational invariance $\leftrightarrow$ Conservation of angular momentum

In spherical coordinates,

(i) the gradient operator is

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} .$$

(ii) the Laplacian is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2} .$$

(iii) the angular momentum operators are

$$
\begin{aligned}
L_x &= i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) \\
L_y &= i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) \\
L_z &= -i\hbar\frac{\partial}{\partial\phi}\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
L_x^2 = (i\hbar)^2 \quad & (\sin^2\phi\frac{\partial^2}{\partial\theta^2} - \csc^2\theta\sin\phi\cos\phi\frac{\partial}{\partial\phi} + \cot\theta\sin\phi\cos\phi\frac{\partial^2}{\partial\theta\partial\phi} \\
& + \cot\theta\cos^2\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\cos\phi\frac{\partial^2}{\partial\theta\partial\phi} \\
& - \cot\theta^2\sin\phi\cos\phi\frac{\partial}{\partial\phi} + \cot\theta^2\cos^2\phi\frac{\partial}{\partial\phi})
\end{aligned}
$$

$$
\begin{aligned}
L_y^2 &= (i\hbar)^2 \quad (\cos^2\phi \frac{\partial^2}{\partial\theta^2} + \csc^2\theta \sin\phi\cos\phi \frac{\partial}{\partial\phi} - \cot\theta\sin\phi\cos\phi \frac{\partial^2}{\partial\theta\partial\phi} \\
&\qquad + \cot\theta \sin^2\phi \frac{\partial}{\partial\theta} - \cot\theta\sin\phi\cos\phi \frac{\partial^2}{\partial\theta\partial\phi} \\
&\qquad + \cot\theta^2 \sin\phi\cos\phi \frac{\partial}{\partial\phi} + \cot^2\theta\sin^2\phi \frac{\partial}{\partial\phi})
\end{aligned}
$$

$$
\begin{aligned}
L_z^2 &= (-i\hbar)^2 \frac{\partial^2}{\partial\phi^2} \quad \text{and} \\
L^2 &= L_x^2 + L_y^2 + L_z^2 \\
&= (i\hbar)^2 (\frac{\partial^2}{\partial\theta^2} + \cot\theta \frac{\partial}{\partial\theta} + \csc^2\theta \frac{\partial^2}{\partial\phi^2}) \\
&= (i\hbar)^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta \frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right] \\
&= -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta \frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right] .
\end{aligned}
$$

Thus we can write the Laplacian as

$$\begin{aligned}\nabla^2 &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{L^2}{(i\hbar)^2} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{L^2}{\hbar^2 r^2}\end{aligned}$$

And the Schrödinger equation for this system becomes

$$H\psi(r,\theta,\phi) = E\psi(r,\theta,\phi)$$

$$\begin{aligned}[-\frac{\hbar^2}{2m}\nabla^2 + V(r)]\psi(r,\theta,\phi) &= E\psi(r,\theta,\phi) \\ \{-\frac{\hbar^2}{2m}[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) - \frac{L^2}{\hbar^2 r^2}] + V(r)\}\psi(r,\theta,\phi) &= E\psi(r,\theta,\phi)\end{aligned}$$

Let us now use a separable solution

$$\psi(r,\theta,\phi) = R(r)F(\theta,\phi)$$

The equation now becomes

$$F(\theta,\phi)\{-\frac{\hbar^2}{2m}[\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d}{dr})R(r)+[V(r)-E]R(r)\}=-R(r)\frac{L^2}{2mr^2}F(\theta,\phi)$$

or

$$\frac{1}{R(r)}\{\frac{d}{dr}(r^2\frac{d}{dr})R(r)+\frac{2mr^2}{\hbar^2}[E-V(r)]R(r)\}=\frac{1}{F(\theta,\phi)}\frac{L^2}{\hbar^2}F(\theta,\phi)=\lambda$$

Thus we have

$$\frac{d}{dr}(r^2\frac{dR}{dr})+\frac{2mr^2}{\hbar^2}[E-V(r)]R(r)=\lambda R(r)$$

and

$$L^2F(\theta,\phi)=\hbar^2\lambda F(\theta,\phi)$$

This shows that $\hbar^2\lambda$ is the eigenvalue of the operator L^2 with the eigenfunction $F(\theta,\phi)$. Furthermore, the solution of the radial equation depends on the form of the potential.

Let's consider the angular part,

$$L^2 F(\theta, \phi) = \hbar^2 \lambda F(\theta, \phi)$$

or

$$-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] F(\theta,\phi) = \lambda F(\theta,\phi)\,.$$

Let us further separate variables with

$$F(\theta, \phi) = \Theta(\theta)\Phi(\phi).$$

Then the equation becomes

$$\Phi(\phi)\left[\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + \lambda\Theta\right] = -\frac{\Theta(\theta)}{\sin^2\theta}\frac{d^2\Phi}{d\phi^2}$$

or

$$\frac{1}{\Theta(\theta)}\left[\sin\theta\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + \lambda\sin^2\theta\right]\Theta = -\frac{1}{\Phi(\phi)}\frac{d^2\Phi}{d\phi^2} = \alpha.$$

Thus we have

$$\sin\theta \frac{d}{d\theta}(\sin\theta \frac{d\Theta}{d\theta}) + \lambda \sin^2\theta\Theta = \alpha\Theta$$

and

$$\frac{L_z^2}{(i\hbar)^2}\Phi = \frac{d^2\Phi}{d\phi^2} = -\alpha\Phi(\phi).$$

The ϕ−equation

The ϕ−equation now becomes

$$\frac{L_z^2}{(i\hbar)^2}\Phi = \frac{d^2\Phi}{d\phi^2} = -\alpha\Phi(\phi)$$

and the solution is

$$\Phi(\phi) = Ae^{i\sqrt{\alpha}\phi} + Be^{-i\sqrt{\alpha}\phi}.$$

Since the wave function has to be single valued and continuous, we expect

$$\alpha = m^2$$

where $m = 0, \pm 1, \pm 2, \cdots =$ integers.

Thus the normalized eigenfunction in the ϕ basis is

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}, \;\; 0 \leq \phi \leq 2\pi.$$

It is also clear that $m\hbar$ is the eigenvalue of L_z with the eigenfunction $\Phi_m(\phi)$. The integer nature of m arises because we want

$$\Phi_m(\phi) = \Phi_m(\phi + 2\pi).$$

The $\theta-$equation

The $\theta-$equation now becomes

$$\sin\theta\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta})+\lambda\sin^2\theta\Theta=m^2\Theta$$

or

$$\sin\theta\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta})+(\lambda\sin^2\theta-m^2)\Theta=0.$$

Let $x=\cos\theta$, then

$$\frac{d}{d\theta}=\frac{dx}{d\theta}\frac{d}{dx}=-\sin\theta\frac{d}{dx}=-(1-x^2)^{1/2}\frac{d}{dx}.$$

The $\theta-$equation becomes

$$(1-x^2)\frac{d}{dx}[(1-x^2)\frac{d\Theta}{dx}]+[\lambda(1-x^2)-m^2]\Theta=0\,.$$

That leads to

$$(1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \left[\lambda - \frac{m^2}{(1-x^2)}\right]\Theta = 0\,.$$

We expect the solution of the following form

$$\Theta(x) = (1-x^2)^{\frac{|m|}{2}} \sum_{k=0}^{\infty} a_k x^k = (1-x^2)^{\frac{|m|}{2}} z(x) \quad \text{with} \quad z(x) = \sum_{k=0}^{\infty} a_k x^k\,.$$

Putting this back into the equation, we obtain

$$\begin{aligned}
&(1-x^2)\Bigg[(1-x^2)^{\frac{|m|}{2}}\frac{d^2 z}{dx^2} - 2|m|x(1-x^2)^{\frac{|m|}{2}-1}\frac{dz}{dx} \\
&+\left(-|m|(1-x^2)^{\frac{|m|}{2}-1} + |m|(|m|-2)x^2(1-x^2)^{\frac{|m|}{2}-2}\right) z\Bigg] \\
&-2x\left[(1-x^2)^{\frac{|m|}{2}}\frac{dz}{dx} - |m|x(1-x^2)^{\frac{|m|}{2}-1} z\right] \\
&+\left[\lambda - \frac{m^2}{(1-x^2)}\right](1-x^2)^{\frac{|m|}{2}} z(x) = 0\,.
\end{aligned}$$

That leads to

$$\left[(1-x^2)\frac{d^2z}{dx^2} - 2(|m|+1)x\frac{dz}{dx} + (\lambda - |m|)z\right](1-x^2)^{\frac{|m|}{2}}$$

$$+(1-x^2)^{\frac{|m|}{2}-1}\left[+2|m|(\frac{|m|}{2}-1)x^2 + 2|m|x^2 - m^2\right]z = 0$$

or

$$(1-x^2)\frac{d^2z}{dx^2} - 2(|m|+1)x\frac{dz}{dx} + \Big[\lambda - |m|(|m|+1)\Big]z = 0\,.$$

Applying the power series solution for z, we obtain

$$(1-x^2)\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - 2(|m|+1)x\sum_{k=1}^{\infty} k a_k x^{k-1}$$

$$+\Big[\lambda - |m|(|m|+1)\Big]\sum_{k=0}^{\infty} a_k x^k = 0 \quad \text{with} \quad z(x) = \sum_{k=0}^{\infty} a_k x^k\,.$$

That leads to

$$\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - \sum_{k=2}^{\infty} k(k-1)a_k x^k - 2(|m|+1)\sum_{k=1}^{\infty} k a_k x^k$$

$$+[\lambda - |m|(|m|+1)]\sum_{k=0}^{\infty} a_k x^k = 0\,, \quad \text{or}$$

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=0}^{\infty} k(k-1)a_k x^k - 2(|m|+1)\sum_{k=0}^{\infty} k a_k x^k$$

$$+[\lambda - |m|(|m|+1)]\sum_{k=0}^{\infty} a_k x^k = 0\,, \quad \text{or}$$

$$\sum_{k=0}^{\infty} x^k \Big[(k+2)(k+1)a_{k+2}$$

$$-\Big(k(k-1) + 2k(|m|+1) + |m|(|m|+1) - \lambda\Big)a_k\Big] = 0\,.$$

Then we have

$$
\begin{aligned}
a_{k+2} &= \frac{1}{(k+2)(k+1)}\Big[k^2 + k(2|m|+1) + |m|(|m|+1) - \lambda\Big]a_k \\
&= \frac{1}{(k+2)(k+1)}\Big[(k+|m|)(k+|m|+1) - \lambda\Big]a_k \,.
\end{aligned}
$$

This defines the recursion relation for the power series solution. Clearly for large k

$$
a_{k+2} \simeq a_k \quad \text{and} \quad z(x) \sim \frac{1}{1-x^2} = 1 + x^2 + x^4 + \mathcal{O}(x^6)\,.
$$

This would imply that the solution blows up for some value of m.

For a physical solution to exist, the series must terminate and we have

$$
\begin{gathered}
(k+|m|)(k+|m|+1) - \lambda = 0\,, \quad \text{or} \\
\lambda = \ell(\ell+1)\,, \quad \text{where} \quad \ell = k + |m|\,.
\end{gathered}
$$

Since both k and m are integers, ℓ also takes integral values.

Furthermore, k and $|m|$ are both positive, then

$$\ell = 0, 1, 2, 3, \cdots$$

and for each ℓ, the integer m takes $2\ell + 1$ values

$$-\ell \leq m \leq \ell$$

Thus the eigenvalues of L^2 are

$$\lambda\hbar^2 = \ell(\ell + 1)\hbar^2$$

where $\ell = 0, 1, 2, 3, \cdots$ and the eigenvalues of L_z are

$$m\hbar$$

where m is an integer and $-\ell \leq m \leq \ell$.

- The recursion relation of a_k implies that if $k = \ell - |m|$ is even the solution contains only even powers of x. However, if k is odd then only the odd terms in the series survive.
- This is similar to the harmonic oscillator solutions.

The solution now depends on two quantum numbers ℓ and m and is denoted by

$$\begin{aligned} z(x) &= z_{\ell,m}(x) \\ x &= \cos\theta. \end{aligned}$$

This is a polynomial of order $k = \ell - |m|$, and the $\theta-$solution

$$\Theta_{\ell,m} = (1 - x^2)^{\frac{|m|}{2}} z_{\ell,m}(x)$$

satisfies the equation

$$(1 - x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + [\ell(\ell+1) - \frac{m^2}{1 - x^2}]\Theta_{\ell,m} = 0.$$

For $m = 0$, the equation is known as the Legendre equation and the solutions of the equation are known as the Legendre functions $P_\ell(x)$:

$$(1 - x^2)\frac{d^2 P_\ell}{dx^2} - 2x\frac{dP_\ell}{dx} + \ell(\ell+1)P_\ell(x) = 0.$$

The Legendre functions (P_ℓ) are polynomials of order ℓ.

The $\Theta_{\ell,m}-$functions are related to the Legendre functions by

$$\Theta_{\ell,m}(x) = (1-x^2)^{\frac{|m|}{2}} \frac{d^{|m|} P_\ell(x)}{dx^{|m|}} = P_{\ell,m}(x)$$

for $\ell \geq |m|$, and are known as the associated Legendre functions.

The complete angular part of the solution is

$$F_{\ell,m}(\theta,\phi) = Y_{\ell,m}(\theta,\phi) = \frac{N_{\ell,m}}{\sqrt{2\pi}} P_{\ell,m}(\theta) e^{im\phi}$$

here

- $N_{\ell,m}$'s are normalization constants, and
- $Y_{\ell,m}$'s are called the spherical Harmonics.

The orthonormal relations for the spherical harmonics can be derived from the orthonormal relations of the eigenvectors of L^2 and L_3:

$$\langle \ell', m' | \ell, m \rangle = \delta_{\ell' \ell} \delta_{m' m} \quad \text{with} \quad Y_{\ell,m}(\theta, \phi) \equiv \langle \theta, \phi | \ell, m \rangle$$

and the completeness relation

$$\int |\theta, \phi\rangle\langle\theta, \phi| \, d\Omega = \int |\theta, \phi\rangle\langle\theta, \phi| \, \sin\theta d\theta \, d\phi = \mathrm{I}\,.$$

That leads to

$$\begin{aligned}
\langle \ell', m' | \ell, m \rangle &= \int \langle \ell', m' | \theta, \phi\rangle\langle\theta, \phi | \ell, m\rangle d\Omega \;=\; \delta_{\ell' \ell} \delta_{m' m} \\
&= \int Y^*_{\ell',m'}(\theta, \phi) Y_{\ell,m}(\theta, \phi) \sin\theta d\theta d\phi \\
&= \frac{N^*_{\ell',m'} N_{\ell,m}}{2\pi} \int P_{\ell',m'} P_{\ell,m} e^{-i(m'-m)\phi} d(\cos\theta) d\phi\,.
\end{aligned}$$

This integral vanishes unless the m quantum numbers are equal. In addition, the integral vanishes unless $\ell' = \ell$.

To see this let us remember that $P_{\ell,m}$ satisfies the equation

$$(1-x^2)\frac{d^2P_{\ell,m}}{dx^2} - 2x\frac{dP_{\ell,m}}{dx} + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P_{\ell,m} = 0$$

or

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_{\ell,m}}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P_{\ell,m} = 0$$

where $P_{\ell,m} = \Theta(\ell,m)$.

We can show that

$$\int [P_{\ell,m}(x)]^2\,dx = \frac{2}{2\ell+1}\frac{(\ell+|m|)!}{(\ell-|m|)!} \quad \text{with} \quad x = \cos\theta\,.$$

Thus our normalization condition now becomes

$$\begin{aligned}\int Y^*_{\ell,m}Y_{\ell,m}\sin\theta d\theta d\phi &= |N_{\ell,m}|^2\int dx[P_{\ell,m}]^2dx \\ &= |N_{\ell,m}|^2\frac{2}{2\ell+1}\frac{(\ell+|m|)!}{(\ell-|m|)!} \;=\; 1\end{aligned}$$

Thus the normalization constant is determined to be

$$N_{\ell,m} = N^*_{\ell,m} = \pm\sqrt{\frac{2\ell+1}{2}\frac{(\ell-|m|)!}{(\ell+|m|)!}}\,.$$

Conventionally we choose the sign to be $(-1)^m$ for $m > 0$ and $+$ for $m \leq 0$. Therefore, the normalized angular solutions are

$$Y_{\ell,m} = \epsilon\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}P_{\ell,m}(\cos\theta)e^{im\phi}$$

where $\epsilon = (-1)^m$ for $m > 0$ and $\epsilon = +1$ for $m \leq 0$.

The complete solution to the Schrödinger equation is

$$\psi_{E,\ell,m}(r,\theta,\phi) = R(r)Y_{\ell,m}(\theta,\phi)\,.$$

The radial part $R(r)$ is determined by the dynamics of the system.