PHYS 3803: Quantum Mechanics I, Spring 2021

Lecture 21, April 15, 2021 (Thursday)

- Reading: Angular Momentum: Griffiths 4.1 and 4.3
- Assignments: Problem Set 9 due April 16 (Friday).
  Submit your homework assignments to Canvas.

Topics for Today: Rotations and Angular Momentum [Griffiths 4.3]

5.2 Rotations and Angular Momentum

5.3 Schrödinger equation for spherically symmetric potentials

Topics for Next Lecture: Angular Momentum

5.3 Schrödinger equation for spherically symmetric potentials

5 Rotations and Angular Momentum

5.2 Rotations and Angular Momentum

To use a more compact notation, let us define

$X = X_1,\ \ Y = X_2,\ \ Z = X_3\,, \quad \text{and} \quad P_x \ = \ P_1,\ \ P_y = P_2,\ \ P_z = P_3\,.$

Thus we can define the angular momentum operator as

$L_i = \epsilon_{ijk} X_j P_k,\ \ i,j,k = 1,2,3 \quad \text{with} \quad \epsilon_{123} = 1,\ \ \epsilon_{213} = -1,\ \ \epsilon_{iik} = 0\,,$

where ϵ_{ijk} is the anti-symmetric Levi-Civita symbol.

Then we have

$$
\begin{aligned}
[L_i, X_j] &= [\epsilon_{ik\ell} X_k P_\ell, X_j] = i\hbar \epsilon_{ijk} X_k\,, \\
[L_i, P_j] &= i\hbar \epsilon_{ijk} P_k\,, \quad \text{and} \\
[L_i, L_j] &= i\hbar \epsilon_{ijk} L_k\,.
\end{aligned}
$$

To study the eigenvalue spectrum of these operators, we further define

$$L_+ \equiv L_1 + iL_2 \, , \quad L_- \equiv L_1 - iL_2 \, , \quad L_- = (L_+)^\dagger \, .$$

The commutation relations of angular momentum operators L_-, L_+, and L_3 are similar to those of $a, a^\dagger$ and H for the harmonic oscillator:

$$\begin{aligned} [L_+, L_3] &= [L_1 + iL_2, L_3] = -\hbar L_+ \, , \\ [L_-, L_3] &= \hbar L_- \, , \quad \text{and} \\ [L_+, L_-] &= 2\hbar L_3 \, . \end{aligned}$$

We know that for a rotationally invariant theory the Hamiltonian commutes with all components of the angular momentum operator. Thus

$$[L_i, H] = 0 \, , \quad \text{and} \quad [L_+, H] = 0 = [L_-, H] \, .$$

We often choose L_3 as the diagonal L_i operator that shares common eigenvectors with L^2 and the Hamiltonian H with

$$[L_3, L^2] = 0\,, \quad [L_3, H] = 0\,, \quad \text{and} \quad [L^2, H] = 0\,.$$

Let $|\lambda, \mu\rangle$ represent the simultaneous eigenstates of the operators L^2 and L_3 such that

$$\begin{aligned} L_3|\lambda,\mu\rangle &= \mu|\lambda,\mu\rangle \ \text{ and} \\ L^2|\lambda,\mu\rangle &= \Lambda|\lambda,\mu\rangle\,. \end{aligned}$$

Let us now examine the effect of the operator L_+ on a given state,

$$\begin{aligned} L_3 L_+|\lambda,\mu\rangle &= (L_+L_3 - [L_+, L_3])|\lambda,\mu\rangle \quad \text{with} \quad [L_+, L_3] = -\hbar L_+ \\ &= (\hbar L_+ + L_+ L_3)|\lambda,\mu\rangle \\ &= (\mu + \hbar) L_+|\lambda,\mu\rangle\,, \quad \text{where} \\ L_3|\lambda,\mu''\rangle &= (\mu+\hbar)|\lambda,\mu''\rangle \quad \text{with} \quad |\lambda,\mu''\rangle \equiv L_+|\lambda,\mu\rangle\,. \end{aligned}$$

In addition, we have

$$\begin{aligned} L^2 L_+|\lambda,\mu\rangle &= ([L^2, L_+] + L_+ L^2)|\lambda,\mu\rangle \\ &= L_+ L^2|\lambda,\mu\rangle \\ &= \Lambda L_+|\lambda,\mu\rangle \,. \end{aligned}$$

We see that the effect of L_+ acting on a state is to raise its eigenvalue μ by a unit of $\hbar$ while leaving the eigenvalue of L^2 unchanged.

Thus we must have

$$L_+|\lambda,\mu\rangle = d_m|\lambda,\mu+\hbar\rangle$$

where d_m are constants depending on λ and m.

We can also show that

$$\begin{aligned}
L_3 L_- |\lambda, \mu\rangle &= (L_- L_3 - [L_-, L_3])|\lambda, \mu\rangle \quad \text{with} \quad [L_-, L_3] = \hbar L_- \\
&= (-\hbar L_- + L_- L_3)|\lambda, \mu\rangle \\
&= (\mu - \hbar) L_- |\lambda, \mu\rangle \,, \quad \text{where} \\
L_3 |\lambda, \mu'\rangle &= (\mu - \hbar)|\lambda, \mu'\rangle \quad \text{with} \quad |\lambda, \mu'\rangle \equiv L_- |\lambda, \mu\rangle \,.
\end{aligned}$$

and

$$\begin{aligned}
L^2 L_- |\lambda, \mu\rangle &= ([L^2, L_-] + L_- L^2)|\lambda, \mu\rangle \\
&= L_- L^2 |\lambda, \mu\rangle \;=\; \Lambda L_- |\lambda, \mu\rangle \,.
\end{aligned}$$

Here we notice that the operator L_- decrease the eigenvalue of L_3 by a unit of $\hbar$ while leaving the eigenvalue of L^2 unchanged.

Thus we expect

$$L_- |\lambda, \mu\rangle = c_m |\lambda, \mu - \hbar\rangle$$

where c_m are constants depending on λ and μ.

- Since the operators L_+ and L_- raise and lower the eigenvalue of L_3, they are also known as the raising and lowering operators.
- Given a state $|\lambda, \mu\rangle$ we can construct a sequence of states $|\lambda, \mu + \hbar\rangle$, $|\lambda, \mu + 2\hbar\rangle$, $\cdots$, and $|\lambda, \mu - \hbar\rangle$, $|\lambda, \mu - 2\hbar\rangle$, $\cdots$, respectively by applying the raising and lowering operators.
- However, physically this sequence cannot go on without termination. For the operator

$$L^2 \quad = \quad L_1^2 + L_2^2 + L_3^2$$

we have

$$L^2 - L_3^2 \quad = \quad L_1^2 + L_2^2 \geq 0.$$

This is a positive semi-definite operator. Thus the eigenvalues must satisfy

$$\Lambda - \mu^2 \geq 0 \quad \text{or} \quad \Lambda \geq \mu^2 \,.$$

This implies that there must exist states with a maximum μ such that

$$\begin{aligned} L_+|\lambda,\mu_{\rm max}\rangle &= 0 \quad \text{and} \\ \langle\lambda,\mu_{\rm max}|L_-L_+|\lambda,\mu_{\rm max}\rangle &= \langle\lambda,\mu_{\rm max}|(L^2-L_3^2-\hbar L_3)|\lambda,\mu_{\rm max}\rangle \\ &= (\Lambda-\mu_{\rm max}^2-\hbar\mu_{\rm max})\langle\lambda,\mu_{\rm max}|\lambda,\mu_{\rm max}\rangle \\ &= \Lambda-\mu_{\rm max}(\mu_{\rm max}+\hbar) \;=\; 0\,, \quad \text{where} \\ L_-L_+=(L_1-iL_2)(L_1+iL_2) &= L_1^2+L_2^2+i[L_1,L_2]=L_1^2+L_2^2-\hbar L_3\,. \end{aligned}$$

Similarly, we can show that there must also exist a state with a minimum μ such that

$$\begin{aligned} L_-|\lambda,\mu_{\rm min}\rangle &= 0 \quad \text{and} \\ \langle\lambda,\mu_{\rm min}|L_+L_-|\lambda,\mu_{\rm min}\rangle &= \langle\lambda,\mu_{\rm min}|(L^2-L_3^2+\hbar L_3)|\lambda,\mu_{\rm min}\rangle \\ &= (\Lambda-\mu_{\rm min}^2+\hbar\mu_{\rm min})\langle\lambda,\mu_{\rm min}|\lambda,\mu_{\rm min}\rangle \\ &= \Lambda-\mu_{\rm min}(\mu_{\rm min}-\hbar) \;=\; 0\,, \quad \text{where} \\ L_+L_-=(L_1+iL_2)(L_1-iL_2) &= L_1^2+L_2^2-i[L_1,L_2]=L_1^2+L_2^2+\hbar L_3\,. \end{aligned}$$

Comparing the two relations, we obtain
$\Lambda - \mu_{\rm max}(\mu_{\rm max} + \hbar) = \Lambda - \mu_{\rm min}(\mu_{\rm min} - \hbar) = 0.$
That leads to

$$\mu_{\rm min} = -\mu_{\rm max} \,.$$

N.B. The other solution has $\mu_{\rm max} = \mu_{\rm min} - \hbar$ that is not meaningful.

Furthermore, let us assume that we can go from the state $|\lambda, \mu_{\rm min}\rangle$ to $|\lambda, \mu_{\rm max}\rangle$ by applying the operator L_+, k times. Thus

$$\mu_{\rm max} - \mu_{\rm min} \;\; = \;\; k\hbar \,, \quad 2\mu_{\rm max} = k\hbar \,, \quad \text{and} \quad \mu_{\rm max} = \frac{k}{2}\hbar = -\mu_{\rm min} \,.$$

Then

$$\begin{aligned}
\Lambda &= \mu_{\rm max}(\mu_{\rm max} + \hbar) \\
&= \frac{k}{2}\hbar\left(\frac{k}{2}\hbar + \hbar\right) \\
&= \left[\frac{k}{2}\left(\frac{k}{2} + 1\right)\right]\hbar^2 \,.
\end{aligned}$$

We may define $\ell = k/2$ that takes only multiples of half integral values. Thus we have

$$\Lambda = \ell(\ell+1)\hbar^2$$

and

$$-\ell\hbar \leq m\hbar \leq \ell\hbar \quad \text{or} \quad -\ell \leq m \leq \ell$$

where m takes $2\ell + 1$ values and ℓ takes values

$$\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$$

that has positive multiple of half integers.

Let us define m to be a number $-\ell \leq m \leq \ell$ and we can determine the normalized states as eigenvectors

$$\begin{aligned} L^2|\ell, m\rangle &= \ell(\ell+1)\hbar^2|\ell, m\rangle \\ L_3|\ell, m\rangle &= m\hbar|\ell, m\rangle \,. \end{aligned}$$