PHYS 3803: Quantum Mechanics I, Spring 2021

Lecture 20, April 13, 2021 (Tuesday)

- Reading: Angular Momentum: Griffiths 4.1 and 4.3
- Assignments: Problem Set 9 due April 16 (Friday).
  Submit your homework assignments to Canvas.

Topics for Today: Rotations and Angular Momentum [Griffiths 4.3]

5.2 Rotations and Angular Momentum

5.3 Schrödinger equation for spherically symmetric potentials

Topics for Next Lecture: Angular Momentum

5.3 Schrödinger equation for spherically symmetric potentials

5 Rotations and Angular Momentum

5.2 Rotations and Angular Momentum

Let us generalize the results of two dimensions to three dimensions. There are three generators of infinitesimal rotations in the 3-dimensional space. Let us denote them by

$$L_x = YP_z - ZP_y\,, \quad L_y = ZP_x - XP_z\,, \quad L_z = XP_y - YP_x.$$

Let us find various commutators with $[AB, C] = A[B, C] + [A, C]B$,

$$\begin{aligned}
[L_x, X] &= [YP_z - ZP_y, X] = 0\,, \\
[L_y, X] &= [ZP_x - XP_z, X] = Z\,[P_x, X] = -i\hbar Z\,, \\
[L_z, X] &= [XP_y - YP_x, X] = -Y\,[P_x, X] = i\hbar Y\,.
\end{aligned}$$

To use a more compact notation, let us define

$$x = x_1, \;\; y = x_2, \;\; z = x_3\,, \quad \text{and} \quad p_x = p_1, \;\; p_y = p_2, \;\; p_z = p_3\,;$$
$$X = X_1, \;\; Y = X_2, \;\; Z = X_3\,, \quad \text{and} \quad P_x = P_1, \;\; P_y = P_2, \;\; P_z = P_3\,.$$

Thus we can define the angular momentum operator as

$$\begin{aligned} L_i &= \epsilon_{ijk} X_j P_k, \;\; i,j,k = 1,2,3 \;\text{ and} \\ \epsilon_{123} &= 1, \;\; \epsilon_{213} = -1, \;\; \epsilon_{iik} = 0. \end{aligned}$$

where ϵ_{ijk} is the anti-symmetric Levi-Civita symbol.

Clearly, then

$$\begin{aligned} [L_i, X_j] &= [\epsilon_{ik\ell} X_k P_\ell, X_j] \\ &= \epsilon_{ik\ell} X_k (-i\hbar \delta_{\ell j}) \\ &= (-i\hbar) \epsilon_{ikj} X_k \\ &= i\hbar \epsilon_{ijk} X_k\,. \end{aligned}$$

Homework:

Similarly we can show that

$$[L_i, P_j] = (i\hbar)\epsilon_{ijk} P_k \, .$$

Furthermore, the commutation relation of two angular momentum operators is now

$$[L_i, L_j] = i\hbar\epsilon_{ijk} L_k \, .$$

We will need to apply

(i) $\epsilon_{ijk}\epsilon_{\ell mk} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$.

(ii) ϵ_{ijk} is anti-symmetric.

(iii) Repeated indices are summed.

This shows that generators of angular momentum along different directions do not commute. However

$$[L_i, L_i] = 0 \, , \quad \text{for any } \; i \, .$$

Defining another operator

$$L^2 = \sum_i L_i L_i$$

we have

$$\begin{aligned} [L_i, L^2] &= [L_i, L_j L_j] \\ &= L_j [L_i, L_j] + [L_i, L_j] L_j \\ &= L_j (i\hbar \epsilon_{ijk} L_k) + (i\hbar \epsilon_{ijk} L_k) L_j \\ &= i\hbar \epsilon_{ijk} (L_j L_k + L_k L_j) \\ &= 0\,. \end{aligned}$$

The operator L^2 commutes with all three generators $(L_i, i = 1, 2, 3)$ of infinitesimal rotation.

A theory is rotationally invariant if the generators (L_i and L^2) commute with the Hamiltonian

$$[L_i, H] = 0 \quad \text{and} \quad [L^2, H] = 0\,.$$

There are several things to note.

- Different components of the angular momentum operator do not commute among themselves: $[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$.
- For a rotationally invariant theory, H, L^2 and one component of the angular momentum can be simultaneously diagonalized.
- A simple example of rotationally invariant theory is

  $$H = \frac{P^2}{2\mu} + V(r) = \frac{P^2}{2\mu} + V(X^2 + Y^2 + Z^2)$$

  where the potential only depends on the radial component.
- We can always choose to diagonalize H, L^2, and L_3 simultaneously. That means they can have common eigenvectors.

To study the eigenvalue spectrum of these operators, we further define

$$L_+ \equiv L_1 + iL_2 \,, \quad L_- \equiv L_1 - iL_2 \,, \quad L_- = (L_+)^\dagger$$

Since L^2 commutes with any component L_i, we have

$$[L_+, L^2] = [L_1 + iL_2, L^2] = 0\,.$$

Similarly

$$[L_-, L^2] = [L_1 - iL_2, L^2] = 0\,.$$

On the other hand,

$$\begin{aligned}
[L_+, L_3] &= [L_1 + iL_2, L_3] \\
&= -i\hbar L_2 + i(i\hbar)L_1 \\
&= -\hbar(L_1 + iL_2) \\
&= -\hbar L_+\,.
\end{aligned}$$

Homework:

Similarly, we can show that

$$[L_-, L_3] = \hbar L_-$$

and

$$[L_+, L_-] = 2\hbar L_3 \, .$$

We know that for a rotationally invariant theory the Hamiltonian commutes with all components of the angular momentum operator. Thus

$$[L_+, H] = [L_-, H] = 0 \, .$$

Let $|\lambda, \mu\rangle$ represent the simultaneous eigenstates of the operators L^2 and L_3 such that

$$\begin{aligned} L_3|\lambda, \mu\rangle &= \mu|\lambda, \mu\rangle \text{ and} \\ L^2|\lambda, \mu\rangle &= \Lambda|\lambda, \mu\rangle \, . \end{aligned}$$

Let us now examine the effect of the operator L_+ on a given state,

$$\begin{aligned} L_3 L_+|\lambda,\mu\rangle &= (L_+L_3 - [L_+,L_3])|\lambda,\mu\rangle \quad \text{with} \quad [L_+,L_3] = -\hbar L_+ \\ &= (\hbar L_+ + L_+L_3)|\lambda,\mu\rangle \\ &= (\mu+\hbar)L_+|\lambda,\mu\rangle\,. \end{aligned}$$

Similarly

$$\begin{aligned} L^2 L_+|\lambda,\mu\rangle &= ([L^2,L_+] + L_+L^2)|\lambda,\mu\rangle \\ &= L_+L^2|\lambda,\mu\rangle \\ &= \Lambda L_+|\lambda,\mu\rangle\,. \end{aligned}$$

We see that the effect of L_+ acting on a state is to raise its eigenvalue μ by a unit of $\hbar$ while leaving the eigenvalue of L^2 unchanged.

Thus we must have

$$L_+|\lambda,\mu\rangle = d_m|\lambda,\mu+\hbar\rangle$$

where d_m are constants depending on λ and m.