PHYS 3803: Quantum Mechanics I, Spring 2021

Lecture 19, April 08, 2021 (Thursday)

- Reading: Angular Momentum: Griffiths 4.1 and 4.3
- Assignments: Problem Set 9 due April 16 (Friday).
  Submit your homework assignments to Canvas.

Topics for Today: Rotations and Angular Momentum [Griffiths 4.3]

5.1 Rotations in Two Dimensions

5.2 Rotations and Angular Momentum

Topics for Next Lecture: Angular Momentum

5.3 Schrödinger equation for spherically symmetric potentials

5 Rotations and Angular Momentum

5.1 Rotations in Two Dimensions

In classical mechanics, if we rotate a position vector $(\vec{r})$ by an angle ϕ about the $z-$axis, then the coordinates of the particle change as

$$\begin{aligned} x &\rightarrow x' = x\cos\phi - y\sin\phi \\ y &\rightarrow y' = x\sin\phi + y\cos\phi \end{aligned}$$

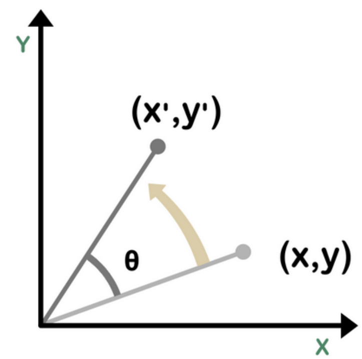

Figure 1: Rotation in the two dimensional (x,y) plane with $\theta \rightarrow \phi$.

Similarly

$$\begin{aligned} p_x &\rightarrow p'_x = p_x \cos\phi - p_y \sin\phi \\ p_y &\rightarrow p'_y = p_x \sin\phi + p_y \cos\phi \end{aligned}$$

We can also write it in the matrix form as

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$\begin{pmatrix} p_x \\ p_y \end{pmatrix} \rightarrow \begin{pmatrix} p'_x \\ p'_y \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} .$$

Let us denote by $R(\phi)$ the matrix that rotates these vectors and $U_R(\phi)$ the operator that acts on the Hilbert space of states corresponding to the rotation $R(\phi)$. Then in the active picture

$$|\psi\rangle \to |\psi_R\rangle = U_R|\psi\rangle \, .$$

To find out the effect of rotation on an arbitrary state, let us examine the effect of rotation on the coordinate basis

$$U_R(\phi)|x, y\rangle = |x\cos\phi - y\sin\phi, x\sin\phi + y\cos\phi\rangle$$

From this again we can show that rotation operator is unitary

$$U_R^\dagger(\phi)U_R(\phi) = I.$$

Let us write the generator for infinitesimal rotation about the $z-$axis as

$$U_R(\epsilon) = I - \frac{i\epsilon}{\hbar}\, G \, .$$

The generators of infinitesimal rotation are are Hermitian because the rotation operators are unitary

$$U_R^\dagger(\phi)U_R(\phi) = I \quad \Longrightarrow \quad U_R^\dagger(\phi) = U_R^{-1}(\phi) \quad \text{and} \quad G^\dagger = G\,.$$

Under infinitesimal rotations with small angle approximation, we have

$$\begin{aligned} \cos\epsilon \quad &\simeq \quad 1 \quad \text{and} \quad \sin\epsilon \simeq \epsilon \\ U_R(\epsilon)|x,y\rangle \quad &= \quad |x-\epsilon y, \epsilon x + y\rangle \quad \text{and} \quad U_R^{-1}(\epsilon)|x,y\rangle = |x+\epsilon y, -\epsilon x + y\rangle \end{aligned}$$

such that $|\psi_R\rangle = U_R(\epsilon)|\psi\rangle$ and

$$\begin{aligned} \psi_R(x,y) \quad &\equiv \quad \langle x,y|\psi_R\rangle \\ &= \quad \langle x,y|U_R(\epsilon)|\psi\rangle \\ &= \quad \langle x+\epsilon y, -\epsilon x + y|\psi\rangle \\ &= \quad \psi(x+\epsilon y, -\epsilon x + y)\,. \end{aligned}$$

We have applied $U_R(\epsilon) = I - (i/\hbar)\epsilon G$,

$$\begin{aligned} \langle x, y | U_R &= [U_R^\dagger | x, y\rangle]^\dagger \; = \; [U_R^{-1} | x, y\rangle]^\dagger \\ &= [|x + \epsilon y, -\epsilon x + y\rangle]^\dagger \; = \; \langle x + \epsilon y, -\epsilon x + y| \,. \end{aligned}$$

That leads to

$$\begin{aligned} \psi_R(x, y) &= \langle x, y | U_R(\epsilon) | \psi\rangle \; = \; \langle x + \epsilon y, -\epsilon x + y | \psi\rangle \\ &= \psi(x + \epsilon y, -\epsilon x + y) \\ &\simeq \psi(x, y) + \epsilon y \frac{\partial}{\partial x}\psi(x, y) - \epsilon x \frac{\partial}{\partial y}\psi(x, y) \quad \text{i.e.} \\ &= \langle x, y | I - \frac{i\epsilon}{\hbar} G | \psi\rangle \; = \; \left(1 - \frac{i}{\hbar}\epsilon G\right)\psi(x, y) \,. \end{aligned}$$

Therefore

$$-\frac{i}{\hbar} G = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \quad \text{and} \quad G = X P_y - Y P_x = L_z \,.$$

The angular momentum operator (L_z) is the generator of infinitesimal rotations about the z-axis.

Furthermore, the theory is rotationally invariant if

$$U^\dagger(R) H U(R) = H \quad \text{(Similarity transformation)}\,.$$

Putting in the infinitesimal structure of $U_R(\epsilon)$, we have

$$\frac{i\epsilon}{\hbar}[L_z, H] = 0 \quad \text{with} \quad U_R(\epsilon) = I - \frac{i}{\hbar}\epsilon L_z$$

or

$$[L_z, H] = 0\,.$$

We can construct a finite rotation about the $z-$axis by taking successive infinitesimal rotations such that $\epsilon = \phi/N, N \to \infty$. Thus we have

$$\begin{aligned} U_R(\phi)) &= \lim_{N\to\infty}(1 - \frac{i\epsilon}{\hbar}L_z)^N \\ &= \lim_{N\to\infty}(1 - \frac{i\phi}{N\hbar}L_z)^N \\ &= e^{-\frac{i\phi}{\hbar}L_z}\,. \end{aligned}$$

Since $[L_z, L_z] = 0$, it is clear that

$$U_R(\phi_1)U_R(\phi_2) = U_R(\phi_1 + \phi_2)\,.$$

That is, rotations about the same axis are additive.

The two dimensional vectors (x, y) can equivalently be described by the circular coordinates (r, ϕ).

- A rotation does not change the radial vector.
- It changes the angle(s).
- Thus in this basis

$$U_R(\Delta\phi)|r, \phi\rangle = |r, \phi + \Delta\phi\rangle\,.$$

Furthermore, note that since $0 \leq \phi \leq 2\pi$, the parameter of rotation is also bounded $0 \leq \phi \leq 2\pi$.

In this basis with polar coordinates (r,ϕ),

$$
\begin{aligned}
|\psi_R\rangle &= U_R|\psi\rangle \\
&= \int r dr d\phi\, U_R(\Delta\phi)|r,\phi\rangle\psi(r,\phi) \\
&= \int r dr d\phi\, |r,\phi+\phi\rangle\psi(r,\phi) \\
&= \int r dr d\phi\, |r,\phi\rangle\psi(r,\phi-\Delta\phi)\,.
\end{aligned}
$$

Thus

$$\langle r,\phi|\psi_R\rangle = \psi(r,\phi-\Delta\phi)$$

or

$$\psi_R(r,\phi) = \psi(r,\phi-\Delta\phi)\,.$$

Furthermore,

$$\psi_R(r,\phi) = \langle r,\phi|U(R)|\psi\rangle = \psi(r,\phi-\Delta\phi)\,.$$

For an infinitesimal rotation $\Delta\phi = \epsilon \to 0+$,

$$\begin{aligned}
\psi_R(r,\phi) &= \langle r,\phi|I - \frac{i\epsilon}{\hbar}L_z|\psi\rangle \\
&= \left(1 - \frac{i\epsilon}{\hbar}L_z\right)\psi(r,\phi) \\
&= \psi(r,\phi-\epsilon) \;=\; \psi(r,\phi) - \epsilon\frac{\partial}{\partial\phi}\psi(r,\phi)\,.
\end{aligned}$$

Thus in the (r,ϕ) basis

$$L_z \to -i\hbar\frac{\partial}{\partial\phi}$$

Furthermore, we can show that rotations form a group. This is a Lie group with transformation operators

$$U_R(\theta) = e^{-(\frac{i}{\hbar})\theta_i L_i} \quad i = 1,2,3$$

where L_i are generators and θ_i are group parameters.

5.2 Rotations and Angular Momentum

Let us generalize the results of two dimensions to three dimensions. There are three generators of infinitesimal rotations in the 3-dimensional space. Let us denote them by

$$\begin{aligned} L_x &= YP_z - ZP_y \\ L_y &= ZP_x - XP_z \\ L_z &= XP_y - YP_x. \end{aligned}$$

Let us find various commutators

$$[L_x, X] = [YP_z - ZP_y, X] = 0$$

$$[L_y, X] = [ZP_x - XP_z, X] = Z[P_x, X] = -i\hbar Z$$

$$[L_z, X] = [XP_y - YP_x, X] = -Y[P_x, X] = i\hbar Y.$$

To use a more compact notation, let us define

$$\begin{aligned} x &= x_1, \; y = x_2, \; z = x_3 \,, \quad \text{and} \quad p_x = p_1, \; p_y = p_2, \; p_z = p_3 \,; \\ X &= X_1, \; Y = X_2, \; Z = X_3 \,, \quad \text{and} \quad P_x = P_1, \; P_y = P_2, \; P_z = P_3 \,. \end{aligned}$$

Thus we can define the angular momentum operator as

$$\begin{aligned} L_i &= \epsilon_{ijk} X_j P_k, \;\; i, j, k = 1, 2, 3 \;\; \text{and} \\ \epsilon_{123} &= 1, \;\; \epsilon_{213} = -1, \;\; \epsilon_{iik} = 0. \end{aligned}$$

where ϵ_{ijk} is the anti-symmetric Levi-Civita symbol.

Clearly, then

$$\begin{aligned} [L_i, X_j] &= [\epsilon_{ik\ell} X_k P_\ell, X_j] \\ &= \epsilon_{ik\ell} X_k (-i\hbar \delta_{\ell j}) \\ &= (-i\hbar) \epsilon_{ikj} X_k \\ &= i\hbar \epsilon_{ijk} X_k \,. \end{aligned}$$

Homework:

Similarly we can show that

$$[L_i, P_j] = (i\hbar)\epsilon_{ijk} P_k \,.$$

Furthermore, the commutation relation of two angular momentum operators is now

$$[L_i, L_j] = i\hbar\epsilon_{ijk} L_k \,.$$

We will need to apply

(i) $\epsilon_{ijk}\epsilon_{\ell mk} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$.

(ii) ϵ_{ijk} is anti-symmetric.

(iii) Repeated indices are summed.

This shows that generators of angular momentum along different directions do not commute. However

$$[L_i, L_i] = 0 \,, \quad \text{for any} \;\; i \,.$$