# PHYS 3803: Quantum Mechanics I, Spring 2021

## Lecture 3, February 02, 2021 (Tuesday)

- Midterm Exam: March 16 (Tuesday), 1:00 p.m.–3:00 p.m.
- Reading: Mathematical Tools (Chapter 3 in Griffiths)
- Assignment: Problem Set 1 due February 03 (Wednesday).
  Make a pdf file and send it to our grader: jmderkacy@ou.edu

## Topics for Today: Review of Classical Mechanics

1.9 Quantum Correspondence Principle

1.10 Postulates of Quantum Mechanics

2.1 Linear Vector Spaces

## Topics for Next Lecture: Mathematical Tools

2.2 Inner Product and Inner Product Spaces

2.3 Dirac Notation

2.4 Linear Operators

2.5 Eigenvectors and Eigenvalues

## 1.9 Quantum Correspondence Principle

The commutation relation of two operators $A$ and $B$ is defined as

$$[A, B] \equiv AB - BA \,.$$

Quantum commutators satisfy the following relations:

(i) $[A, B] = -[B, A]$

(ii) $[A + B, C] = [A, C] + [B, C]$

(iii) $[AB, C] = A[B, C] + [A, C]B$

(iv) $[A, BC] = B[A, C] + [A, B]C$

**Exercise:** Let us consider $X$, $P$, and $H$ as quantum operators, find

- (a) $[X, P^2]$, and
- (b) $[X, H]$, with

$$H = \frac{P^2}{2m} + V(X) \,.$$

**Quantum correspondence principle** is the relation between quantum commutators and Poisson brackets:

$$[\Omega_1, \Omega_2] = i\hbar\{\omega_1, \omega_2\}\,.$$

That means, the commutation relation of two quantum operators is $i\hbar$ times the value of the classical Poisson bracket.

- The constant $\hbar = 1.054 \times 10^{-27}$ erg·sec and is called (reduced) Planck constant.
- The Planck constant $\hbar$ measures the non-classical nature of systems. More commonly we say that we recover classical mechanics in the limit $\hbar \to 0$.

**Exercise:** Let us consider $X$, $P$, $X_i$, and $P_j$ as quantum operators.

- show that $[X, P] = i\hbar\{x, p\} = i\hbar$.
- Find $[X_i, P_j]$.

## 1.10 Postulates of Quantum Mechanics

Given classical Hamiltonian mechanics we go to quantum mechanics through the following postulates:

(i) In quantum mechanics the state of a system at a fixed time is denoted by the state vector $|\psi(t)\rangle$ that belongs to a Hilbert space.

(ii) In quantum mechanics, the classical observables $x$ and $p$ are replaced by operators $X$ and $P$ with the commutation relation

$$[X, P] = i\hbar.$$

The eigenvalue equations for $X$ and $P$ are

$$\begin{aligned} X|x\rangle &= x|x\rangle \\ P|p\rangle &= p|p\rangle \end{aligned}$$

where $|x\rangle$ and $|p\rangle$ are eigenvectors, while $x$ and $p$ are eigenvalues.

The operators have the following form in the x-basis

$$\begin{aligned}\langle x|X|y\rangle &= x\delta(x-y) = y\delta(x-y)\,, \\ \langle x|P|y\rangle &= -i\hbar\frac{d}{dx}\delta(x-y)\,,\end{aligned}$$

where $|x\rangle$ is a column vector and $\langle x|$ is a row vector.

Any operator $\Omega$ corresponding to the observable $\omega$ is obtained as the same function of the operators $X$ and $P$. Thus

$$\omega(x,p) \to \Omega(X,P)\,.$$

When dealing with products of two operators, we will symmetrize them. Thus

$$xp \to \frac{1}{2}(XP+PX)\,.$$

(iii) For an operator ($\Omega$) with eigenvalues ($\omega_i$), we have

$$\Omega|\omega_i\rangle = \omega_i|\omega_i\rangle \quad \text{(eigenvalue equation)}.$$

Quantum mechanics gives probabilistic results. If a system is in a state $|\psi\rangle$, then a measurement corresponding to $\Omega$ yields one of the eigenvalues $\omega_i$ of $\Omega$ with a probability

$$P(\omega_i) = \frac{|\langle\omega_i|\psi\rangle|^2}{\langle\psi|\psi\rangle}\,, \quad \sum_i P(\omega_i) = 1\,,$$

where $\langle\omega_i|\psi\rangle$ is the inner product of the column vector $|\psi\rangle$ and the row vector $\langle\omega_i|$.

The result of the measurement would be to change the state of the system to the eigenstate $|\omega_i\rangle$ of the operator $\Omega$.

(iv) In classical mechanics the state variables change according to Hamilton's equations of motion

$$\begin{aligned}\dot{x} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial x}\end{aligned}$$

In quantum mechanics the state vector evolve with time according to Schrödinger equation

$$H|\psi(t)\rangle = E|\psi(t)\rangle \quad \text{with} \quad E = i\hbar\frac{d}{dt}$$

where $H = H(X, P) =$ the Hamiltonian operator.

If $|\psi\rangle$ is an eigenstate with energy $E$ then we can write

$$H|\psi\rangle = E|\psi\rangle \quad \text{(eigenvalue equation)}$$

just as we write $P|p\rangle = p|p\rangle$.

# 2 Mathematical Introduction

## 2.1 Linear Vector Spaces

### Vector

A set of quantities $\{\vec{v}_i\}$ with definite rules for addition and multiplication is called a set of vectors if they satisfy

$$\vec{v}_i + \vec{v}_j = \vec{v}_j + \vec{v}_i, \tag{1}$$

$$\vec{v}_i + (\vec{v}_j + \vec{v}_k) = (\vec{v}_i + \vec{v}_j) + \vec{v}_k, \tag{2}$$

$$\alpha(\vec{v}_i + \vec{v}_j) = \alpha\vec{v}_i + \alpha\vec{v}_j, \tag{3}$$

$$(\alpha + \beta)\vec{v}_i = \alpha\vec{v}_i + \beta\vec{v}_j, \tag{4}$$

$$(\alpha\beta)\vec{v}_i = \alpha(\beta\vec{v}_i). \tag{5}$$

What is the name of each property?

## 2.1 Linear Vector Spaces

### Vector

A set of quantities $\{\vec{v}_i\}$ with definite rules for addition and multiplication is called a set of vectors if they satisfy

$$\begin{aligned}
\vec{v}_i + \vec{v}_j &= \vec{v}_j + \vec{v}_i, \text{ (commutative law of addition)} \\
\vec{v}_i + (\vec{v}_j + \vec{v}_k) &= (\vec{v}_i + \vec{v}_j) + \vec{v}_k, \text{ (associative law of addition)} \\
\alpha(\vec{v}_i + \vec{v}_j) &= \alpha\vec{v}_i + \alpha\vec{v}_j, \text{ (distributivity w.r.t vector addition)} \\
(\alpha + \beta)\vec{v}_i &= \alpha\vec{v}_i + \beta\vec{v}_j, \text{ (distributivity w.r.t number addition)} \\
(\alpha\beta)\vec{v}_i &= \alpha(\beta\vec{v}_i), \text{ (associative law of multiplication)}
\end{aligned}$$

where $\alpha, \beta \in \mathcal{C}$ and w.r.t. = 'with respect to'.

**Linear Vector Space**

If $V$ represents the set of vectors $\{\vec{v}_i\}$ such that

1. $\alpha\vec{v}_i + \beta\vec{v}_j \in V$,
2. there exists a unique null vector or zero vector $\emptyset \in V$ such that $\vec{v}_i + \emptyset = \vec{v}_i = \emptyset + \vec{v}_i$,
3. for every vector $\vec{v}_i$, there exists a unique inverse $-\vec{v}_i \in V$ such that $\vec{v}_i + (-\vec{v}_i) = \emptyset$,

then $V$ is called a linear vector space.

Clearly, the familiar vectors in the 3-dimensional space represent a linear vector space. In that case, addition involves both magnitudes and directions of vectors. The null vector in this case is a vector of zero magnitude and the inverse is a vector with the arrow reversed.

## Linear Independence

A set of vectors $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_N\}$ is said to be linearly independent if a relation of the type

$$\sum_{i=1}^{N} \alpha_i \vec{v}_i = 0\,,$$

has the only solution that all $\alpha_i$'s vanish, $\alpha_i = 0$.

## Dimensionality

A vector space $V$ is said to be $N$ dimensional, and is denoted by $V^N$ if the maximum number of linearly independent vectors that can be found in that space is $N$

## Theorem

An arbitrary nontrivial vector $\vec{v}$ in $V^N$ can be uniquely expressed as a linear combination of $N$ linearly independent vectors in $V^N$.