PHYS 3803: Quantum Mechanics I, Spring 2021

Week 1: January 25 (M)–January 29 (F)

Lecture 1, January 26, 2021 (Tuesday)

- Handouts: Syllabus (Check your OU Email)
- Midterm Exam: March 12 (Friday), 1:30 p.m.–3:30 p.m.
- Final Exam: May 11 (Tuesday), 1:30 p.m.–3:30 p.m.
- Reading: Study Lagrangian and Hamiltonian Dynamics with your favorite Classical Mechanics textbook.
- Assignment: Problem Set 1 due February 03 (Wednesday). Make a pdf file and send it to our grader: jmderkacy@ou.edu

Topics for Today: Review of Classical Mechanics

A The Hamiltonian and Quantum Mechanics

1.1 The Lagrangian, the Action, and Hamilton's Principle

1.2 Functional Derivatives

1.3 Back to Hamilton's Principle

1.4 More Degrees of Freedom

1.5 The Euler-Lagrange Equation

1.6 The Lagrangian and the Sum Over Paths

Topics for Quantum Mechanics I

1 Review of Classical Mechanics [Taylor 7, 13]

2 Mathematical Tools [Griffiths 3]

3 One-dimensional Schrödinger Equation [Griffiths 1 and 2]

4 Harmonic Oscillator [Griffiths 2.3]

5 Angular Momentum [Griffiths 4.1 and 4.3]

6 Hydrogen Atom [Griffiths 4.2]

7 BONUS: Path Integral

8 Symmetries and Conservation Laws [Griffiths 6]

Canonical Momentum and the Hamiltonian

Recall that $T(q, \dot{q}, t)$ = the kinetic energy, $U(q, \dot{q}, t)$ = the potential energy, and the Lagrangian is defined as

$$L(q, \dot{q}, t) \equiv T(q, \dot{q}, t) - U(q, \dot{q}, t)\,.$$

With the conjugate momentum or the canonical momentum $p_i \equiv \partial L/\partial \dot{q}_i$, the Legendre transformation takes $L(q_i, \dot{q}_i, t)$ to $H(q_i, \partial L/\partial \dot{q}_i, t) = H(q_i, p_i, t)$.

The Hamiltonian is defined as

$$H(q, p, t) \equiv \sum_i \dot{q}_i \left(\frac{\partial L}{\partial \dot{q}_i}\right) - L(q, \dot{q}, t) = \sum_i \dot{q}_i p_i - L(q, \dot{q}, t)\,.$$

Physical momentum ($\vec{p} = m\vec{v}$) and canonical momentum ($P_i \equiv \partial L/\partial \dot{q}_i$) are different if the potential energy depends on $\dot{q}_i$.

Formalism in Quantum Mechanics

A. The Hamiltonian and Quantum Mechanics

The most common formalism in Quantum Mechanics is **Canonical Quantization** or **Operator Formalism**.

A quantum system is described with a state vector $|\psi(t)\rangle$ that becomes a wave function $\psi(\vec{x}, t)$ in coordinate basis, and the equation of motion is the Schrödinger equation

$$\begin{aligned} H|\psi(t)\rangle &= E|\psi(t)\rangle \,, \quad \text{(Hilbert space)} \,, \\ H\psi(\vec{x},t) &= E\psi(\vec{x},t) \,, \quad \text{(Coordinate space)} \,, \\ H &= \frac{P^2}{2m} + U \,, \quad P \equiv -i\hbar\nabla \,, \quad \text{and} \quad E \equiv i\hbar\frac{\partial}{\partial t} \,, \end{aligned}$$

where $H =$ Hamiltonian, and
$\psi(\vec{x}, t) =$ the wave function in the coordinate basis.

In addition, the quantum operators of every coordinate and its conjugate momentum satisfy the canonical commutation relation

$$[X, P] \equiv XP - PX = i\hbar,$$

and the uncertainty principle

$$\Delta X \Delta P \geq \frac{\hbar}{2},$$

where $\hbar = h/(2\pi)$, and $h \equiv$ Planck's constant.

Let us consider classical observables as continuous functions of generalized coordinates and conjugate momenta: $\omega(q, p)$, $\omega_1(q, p)$ and $\omega_2(q, p)$ where q and p are vectors with components q_k and p_k, $k = 1, \cdots, N$. The Poisson brackets are defined as

$$\{\omega_1, \omega_2\} \equiv \sum_{k=1}^{N} \left(\frac{\partial \omega_1}{\partial q_k} \frac{\partial \omega_2}{\partial p_k} - \frac{\partial \omega_1}{\partial p_k} \frac{\partial \omega_2}{\partial q_k} \right).$$

Quantum correspondence principle is the relation between quantum commutators and Poisson brackets:

$$[\Omega_1, \Omega_2] = i\hbar\{\omega_1, \omega_2\}\,.$$

That means, the commutation relation of two quantum operators is $i\hbar$ times the value of the classical Poisson bracket.

It is clear at this point that the Planck's constant $\hbar$ measures the non-classical nature of systems. More commonly we say that we recover classical mechanics in the limit $\hbar \to 0$.

Exercise: Let us consider X, P as quantum operators, show that

$$[X, P] = i\hbar\{x, p\} = i\hbar\,.$$

1 Review of Classical Mechanics

1.1 Lagrangian, Action, and Hamilton's Principle

Let us consider one dimensional motion of a particle with coordinate $x(t)$ and velocity $v(t) = \dot{x}(t)$ in a potential $U(x)$. The kinetic energy is

$$T(x, \dot{x}) = \frac{1}{2} m \dot{x}^2$$

and the potential energy is $U(x, \dot{x})$.

In general, the Lagrangian is defined as

$$L(x, \dot{x}) \equiv T(x, \dot{x}) - U(x, \dot{x})\,.$$

For a system with a conservative force, the Lagrangian becomes

$$L(x, \dot{x}) \equiv T(\dot{x}) - V(x)\,.$$

It is a scalar, i.e., it is invariant under Lorentz transformation.

The action (S) of a particle moving along along a trajectory $[x(t)]$ is

$$S[x] \equiv \int_{t_a}^{t_b} L(x, \dot{x})\, dt\,.$$

Hamilton's principle is the statement that the action (S) for a particle moving from x_a at t_a to x_b at t_b

$$S[x] \equiv \int_{t_a}^{t_b} [T(x(t), \dot{x}(t)) - U(x(t), \dot{x}(t))]\, dt$$

along a path $x(t)$ is stationary (minimum or maximum) for the actual classical trajectory $x(t) = x_0(t)$ that follows Newton's law of motion. Note that

(a). $L \equiv T - U \equiv$ the Lagrangian is a function of two variables: x and $\dot{x}$.

(b). $S[x] \equiv$ the action is a function whose argument is itself a function. A function of function is sometimes called a functional.

(c). The action $S[x]$ is stationary for the classical trajectory $x_0(t)$.

Taylor Expansion: Taylor expansion is the most useful mathematical formula for physicists.

- A function can be approximated by polynomials in the neighborhood of a point (x_0) in terms of its value and derivatives using Taylor expansion about x_0 with $f(x) = f(x_0 + \Delta x)$:

$$f(x_0 + \Delta x) = f(x_0) + \Delta x \frac{df}{dx}(x_0) + \mathcal{O}[(\Delta x)^2] = \sum_{n=0}^{\infty} \frac{(\Delta x)^n}{n!} \frac{d^n f}{dx^n}(x_0)\,.$$

- For a function of two variables the Taylor expansion becomes

$$\begin{aligned} f(x,y) &= f(x_0 + \Delta x, y_0 + \Delta y) \\ &= f(x_0, y_0) + \Delta x \frac{\partial f}{\partial x}(x_0, y_0) + \Delta y \frac{\partial f}{\partial y}(x_0, y_0) \\ &\quad + \frac{1}{2}\left[(\Delta x)^2 \frac{\partial^2 f}{\partial x^2} + (\Delta y)^2 \frac{\partial^2 f}{\partial y^2} + 2\Delta x \Delta y \frac{\partial^2 f}{\partial x \partial y}\right] + \mathcal{O}(\Delta^3)\,. \end{aligned}$$

1.2 Functional Derivatives

For the functional

$$W[x] = \int_{t_a}^{t_b} F(x(t), \dot{x}(t))\, dt,$$

the functional derivative of $W[x]$ with respect to $x(t)$ is defined as $\delta W[x]/\delta x(t) \equiv$ the coefficient of the linear term in $\delta x(t)$ in $W[x + \delta x]$.

The functional derive becomes

$$\frac{\delta W[x]}{\delta x(t)} \equiv \lim_{\delta x \to 0} \frac{W[x + \delta x] - W[x]}{\delta x(t)} = \frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right).$$

We can write

$$\frac{\delta W[x]}{\delta x(t)}|_{x=x_0} = \frac{\delta W}{\delta x(t)}[x_0]$$

for the path with $x_0(t)$.

For

$$W[x] = \int_{t_a}^{t_b} [x(t) - t]^2 dt \,,$$

we have

$$\frac{\delta W[x_0]}{\delta x(t)} = 2[x_0(t) - t],$$

where we should have applied a useful identity

$$\frac{\delta x_a(\tau)}{\delta x_b(t)} = \delta_{ab}\delta(\tau - t) \ .$$

In terms of the functional derivative, the functional Taylor series has the following form

$$\begin{aligned} W[x + \delta x] &= W[x] + \int_{t_a}^{t_b} \frac{\delta W[x]}{\delta x(t)} \delta x(t) dt + \cdots \\ &= exp\left(\int_{t_a}^{t_b} \frac{\delta}{\delta x(t)} \right) W[x] \delta x(t) dt \,. \end{aligned}$$

For functionals that have the form of an integral of an ordinary function

$$W[x] = \int_{t_a}^{t_b} F(x)\, dt\,,$$

the functional derivative of $W[x]$ is related to the ordinary derivative dF/dx

$$\frac{\delta W[x]}{\delta x(t)} = \frac{dF(x)}{dx}\,.$$

Here are some examples:

$$(a) \qquad F(x) = x^3(t)\,, \quad W[x] = \int_{t_a}^{t_b} x^3(t) dt$$

$$(b) \qquad F(x) = \sin x(t)\,, \quad W[x] = \int_{t_a}^{t_b} \sin x(t) dt$$

$$(c) \qquad F(x,y) = x^3(t) y^3(t)\,, \quad W[x,y] = \int_{t_a}^{t_b} x^3(t) y^3(t) dt\,.$$

Here are some examples:

$$
\begin{aligned}
(a) \qquad\qquad F(x) &= x^3(t)\,, \quad W[x] = \int_{t_a}^{t_b} x^3(t)dt \\
\frac{\delta W[x]}{\delta x(t)} &= 3x^2(t), \\
(b) \qquad\qquad F(x) &= \sin x(t)\,, \quad W[x] = \int_{t_a}^{t_b} \sin x(t)dt \\
\frac{\delta W[x]}{\delta x(t)} &= \cos x(t)\,, \\
(c) \qquad\qquad F(x,y) &= x^3(t)y^3(t)\,, \quad W[x,y] = \int_{t_a}^{t_b} x^3(t)y^3(t)dt \\
\frac{\delta W[x,y]}{\delta x(t)} &= 3x^2(t)y^3(t)\,.
\end{aligned}
$$

1.3 Back to Hamilton's Principle

The action is defined as

$$\begin{aligned} S[x] &\equiv \int_{t_a}^{t_b} L(x,\dot{x})\,dt \\ &= \int_{t_a}^{t_b} \left(T(x,\dot{x}) - U(x,\dot{x})\right) dt \\ &= S_T - S_U\,, \end{aligned}$$

with

$$S_T[x] = \int_{t_a}^{t_b} T(x,\dot{x})\,dt\,,$$

and

$$S_U[x] = \int_{t_a}^{t_b} U(x,\dot{x})\,dt\,.$$

Then the functional derivative of the action becomes

$$\frac{\delta S[x]}{\delta x(t)} = \frac{\delta S_T[x]}{\delta x(t)} - \frac{\delta S_U[x]}{\delta x(t)}.$$

Let's consider $U = U(x)$ for a conserved energy. Since $U(x)$ is an ordinary function of x,

$$\frac{\delta S_U[x]}{\delta x(t)} = \frac{dU(x)}{dx} = U'(x(t)).$$

The functional derivative of the S_T term can be evaluated from

$$\begin{aligned} S_T[x + \delta x] &= \int_{t_a}^{t_b} \frac{1}{2} m(\dot{x} + \delta\dot{x})^2 \, dt \\ &= \int_{t_a}^{t_b} \frac{1}{2} m\dot{x}^2 \, dt + \int_{t_a}^{t_b} m\dot{x}\delta\dot{x} \, dt + O(\delta x^2) \\ &= S_T[x] + \int_{t_a}^{t_b} m\dot{x}\delta\dot{x} \, dt + O(\delta x^2) \, . \end{aligned}$$

The linear term in $\delta\dot{x}$ becomes

$$\begin{aligned}
\int_{t_a}^{t_b} m\dot{x}\delta\dot{x}\,dt &= \int_{t_a}^{t_b} m\dot{x}\frac{d}{dt}(\delta x)\,dt \\
&= \int_{t_a}^{t_b} m\frac{d}{dt}(\dot{x}\delta x)\,dt - \int_{t_a}^{t_b} m\frac{d}{dt}(\dot{x})\delta x(t)\,dt \\
&= m\dot{x}(t_b)\delta x(t_b) - m\dot{x}(t_a)\delta x(t_a) - \int_{t_a}^{t_b} m\frac{d}{dt}(\dot{x})\delta x(t)\,dt \\
&= m\dot{x}(t_b)\delta x(t_b) - m\dot{x}(t_a)\delta x(t_a) - \int_{t_a}^{t_b} m\ddot{x}\delta x(t)\,dt\,.
\end{aligned}$$

We note that

$$\begin{aligned}
\dot{x} + \delta\dot{x} &= \frac{d}{dt}(x(t) + \delta x(t)) \\
&= \dot{x} + \frac{d}{dt}(\delta x(t)).
\end{aligned}$$

Therefore, we have

$$\delta \dot{x} = \frac{d}{dt}(\delta x(t)).$$

The initial conditions demand

$$\begin{aligned} x(t_a) &= x_a = \text{constant} \\ x(t_b) &= x_b = \text{constant.} \end{aligned}$$

Thus

$$\begin{aligned} \delta x(t_a) &= \delta x_a = 0 \\ \delta x(t_b) &= \delta x_b = 0. \end{aligned}$$

Now we have

$$S_T[x + \delta x] = S_T[x] - \int_{t_a}^{t_b} m\ddot{x}\delta x(t)\, dt$$

and so

$$\frac{\delta S_T[x]}{\delta x(t)} = -m\ddot{x}(t).$$

The extremization condition for $S[x]$ is

$$\begin{aligned} 0 &= \frac{\delta S[x]}{\delta x(t)} \\ &= \frac{\delta S_T[x]}{\delta x(t)} - \frac{\delta S_U[x]}{\delta x(t)} \\ &= -m\ddot{x}(t) - U'(x) \end{aligned}$$

or

$$m\ddot{x}(t) = -U'(x) = F(x),$$

which is just Newton's second law of motion.

1.4 More Degrees of Freedom

Let us consider a system with N particles moving in one dimension with coordinates $x_1, x_2, \cdots, x_N$. The Lagrangian is

$$L = T - U = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{x}_i^2 - U(x_1, x_2, \cdots, x_N)\,,$$

where T is the total kinetic energy and U is the potential energy.

It is easy to see that

$$\frac{\delta S[x]}{\delta x_i(t)} = -m_i \ddot{x}_i - U_i(x_1, \cdots, x_N) = -m_i \ddot{x}_i - \frac{\partial}{\partial x_i} U(x_1, \cdots, x_N) = F_i\,,$$

and $-U_i$ is the force on particle i.

For $S[x]$ to be stationary, we must have

$$\frac{\delta S[x]}{\delta x_i(t)} = -m_i \ddot{x}_i - U_i(x_1, \cdots, x_N) = 0,$$

for each i, which gives $F_i = m\ddot{x}_i = ma_i$ for every particle.

Kronecker delta Symbol and Dirac delta Function

The Kronecker delta symbol (δ_{ij} for discrete variables is defined as

$$\delta_{ij} = 1\,, \quad \text{for } i = j\,, \quad \text{and } \delta_{ij} = 0\,, \quad \text{for } i \neq j\,,$$

such that

$$\sum_{i=1}^{N} \delta_{ij} g_i = g_j\,.$$

The Dirac delta function for continuous variables is defined as

$$\delta(x - x_0) = \infty\,, \quad \text{for } x = x_0\,, \quad \text{and } \delta(x - x_0) = 0\,, \quad \text{for } x \neq x_0\,,$$

such that

$$\int_{x_1}^{x_2} \delta(x - x_0) f(x)\, dx \;=\; f(x_0)\;, \quad \text{for } x_1 < x_0 < x_2\,.$$

Review of Coordinates

Cartesian Coordinates

The displacement of a particle in Cartesian coordinates is described as

$$\vec{r} = \vec{x} = x\hat{x} + y\hat{y} + z\hat{z}$$

where $\hat{x}$, $\hat{y}$, $\hat{y}$ are unit vectors.

Cylindrical Coordinates

The cylindrical coordinates are (ρ, ϕ, z), where

$$\rho = \sqrt{x^2 + y^2}$$

and $0 \leq \phi \leq 2\pi$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right), \quad \text{for } x > 0, \quad \text{and} \quad \phi = -\sin^{-1}\left(\frac{y}{x}\right) + \pi, \quad \text{for } x < 0.$$

Spherical Coordinates

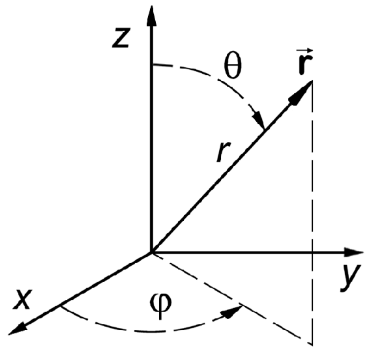

Figure 1: Spherical coordinates

The spherical coordinates are (r, θ, ϕ), where

$$r = \sqrt{x^2 + y^2 + z^2}\,, \quad \text{and} \quad \theta = \cos^{-1}\left(\frac{z}{r}\right)\,,$$

with $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)\,, \quad \text{for } x > 0\,, \quad \text{and} \quad \phi = -\sin^{-1}\left(\frac{y}{x}\right) + \pi\,, \quad \text{for } x < 0\,.$$

1.5 The Euler-Lagrange equation

The action (S) is defined as

$$S[q] \equiv \int_{t_a}^{t_b} L(q, \dot{q}) dt \,.$$

The Lagrangian (L) is defined as the difference of the kinetic energy (T) and the potential energy (U), $L(q, \dot{q}) \equiv T(q, \dot{q}) - U(q, \dot{q})$, and it is a function of generalized coordinates (q_i) and time derivatives of the generalized coordinates ($\dot{q}_i$), where q can be a vector with components $q_i, i = 1, 2, \cdots, N$.

The functional derivative of the action $S[q]$ with respect to $q_i(t)$ is

$$\frac{\delta S[q]}{\delta q_i(t)} = \frac{\partial L(q, \dot{q})}{\partial q_i(t)} - \frac{d}{dt} \left[\frac{\partial L(q, \dot{q})}{\partial \dot{q}_i(t)} \right] \,.$$

The first term arises from the Taylor expansion of the $q_i(t)$ dependence; the second term arises from the Taylor expansion of the $\dot{q}_i(t)$ followed by an integration by parts which gives the minus sign.

In one dimension with $q(t) = x(t)$, we have

$$
\begin{aligned}
S[x+\delta x] &= \int_{t_a}^{t_b} L(x+\delta x, \dot{x}+\delta\dot{x})\,dt \\
&= \int_{t_a}^{t_b} \left[L(x,\dot{x}) + \delta x\frac{\partial L}{\partial x} + \delta\dot{x}\frac{\partial L}{\partial \dot{x}}\right] dt + O(\delta x^2) \\
&= S[x] + \int_{t_a}^{t_b} \left[\delta x\frac{\partial L}{\partial x} + \frac{d}{dt}(\delta x\frac{\partial L}{\partial \dot{x}}) - \delta x\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right] dt + O(\delta x^2) \\
&= S[x] + \int_{t_a}^{t_b} \left[\delta x\frac{\partial L}{\partial x} - \delta x\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right] dt + \left[\delta x\frac{\partial L}{\partial \dot{x}}\right]_{t_a}^{t_b} + O(\delta x^2) \\
&= S[x] + \int_{t_a}^{t_b} \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right] \delta x\,dt + O(\delta x^2),
\end{aligned}
$$

where $\delta x(t_a) = 0 = \delta x(t_b)$ since x_a and x_b are fixed end points.

We have just found

$$S[x+\delta x] = S[x] + \int_{t_a}^{t_b} \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right] \delta x \, d\tau + O(\delta x^2)\,.$$

The functional derivative can be defined as the coefficient of the linear term in $\delta x(t)$: $\quad \delta S[x]/\delta x(t) = \partial L/\partial x(t) - d/dt[\partial L/\partial \dot{x}(t)]$, and it should be derived as

$$\begin{aligned}
\frac{\delta S[x]}{\delta x(t)} &\equiv \lim_{\delta x \to 0} \frac{S[x+\delta x] - S[x]}{\delta x(t)} \\
&= \int_{t_a}^{t_b} \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right] \frac{\delta x(\tau)}{\delta x(t)} d\tau \\
&= \int_{t_a}^{t_b} \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right] \delta(\tau - t) d\tau \\
&= \frac{\partial L}{\partial x(t)} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}(t)}\right].
\end{aligned}$$

If $S[x]$ is stationary, the functional derivative with respect to $x(t)$ must vanish, i.e.

$$\frac{\delta S[x]}{\delta x(t)} = \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0\,.$$

N.B. The Hamilton's principle implies that the solution for the motion along the classical path with stationary action satisfies

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0.$$

which is called the Euler-Lagrange equation. If x or q has several components, the Euler-Lagrange equation must be true for each component (q_i) separately:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] = 0\,.$$

Example: A Frictionless Table

Let's consider a frictionless table in the $x - y$ plane with a hole at the origin. A mass m_1 slides on the surface of the table but it is attached to a massless string of length ℓ that goes through the hole at the center of the table and hangs straight down where it is attached to a mass m_2.

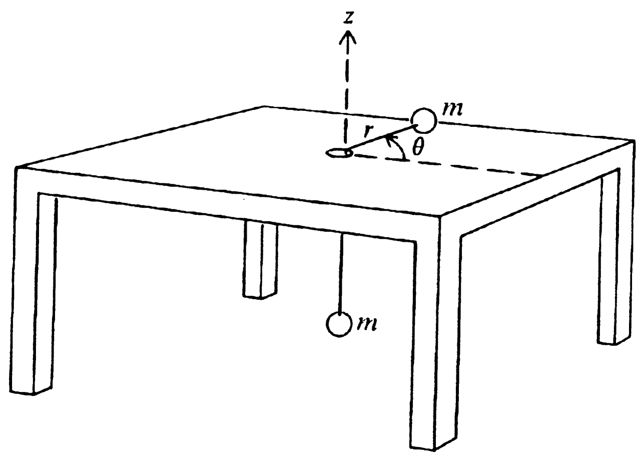

Figure 2: Frictionless table.

This system can be described by the length r of the string on the table and the angle θ of the string on the table from the x axis.

The kinetic energy is

$$T(r,\dot{r},\theta,\dot{\theta}) = \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\theta}^2,$$

where the first term is the kinetic energy of translation and the second term is the kinetic energy of rotation.

The gravitational potential energy is

$$U(r) = m_2 g r,$$

such that $U(r=0) = 0 = U_{\min}$ and $U(r=\ell) = m_2 g\ell = U_{\mathrm{MAX}}$.

The Lagrangian is

$$\begin{aligned} L(r,\dot{r},\theta,\dot{\theta}) &= T - U \\ &= \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\theta}^2 - m_2 g r. \end{aligned}$$

The Euler-Lagrange equations are

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0.$$

The Lagrangian is

$$L(r, \dot{r}, \theta, \dot{\theta}) = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2 \dot{\theta}^2 - m_2 g r \, .$$

The Euler-Lagrange equations are

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0.$$

For $q = r$, the equation of motion is

$$\begin{aligned} \frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} &= 0 \\ m_1 r \dot{\theta}^2 - m_2 g - (m_1 + m_2)\ddot{r} &= 0. \end{aligned}$$

For $q = \theta$, the equation of motion is

$$\begin{aligned} \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= 0 \\ -\frac{d}{dt}(m_1 r^2 \dot{\theta}) &= 0 \, , \quad \text{i.e.} \quad p_\theta \equiv \frac{\partial L}{\partial \dot{\theta}} = m_1 r^2 \dot{\theta} = \text{ constant} \, . \end{aligned}$$