PHYS 6213: Advanced Particle Physics, Spring 2022

Lecture 26, Apr 20, 2022 (Wednesday)

- Reading:
  (a) Chap 12 in Collider Physics
  (b) Chap 18 and Chap 19 in Quantum Field Theory
- Assignments:
  (a) Term Paper 1: $pp \to H + X$ due Apr 28 (Thu)
  (b) Term Paper 2: $pp \to VH + X$ due May 06 (Fri)

Topics for Today:

Chapter 12 Dimensional Regularization and Renormalization

12.5 Renormalization in $\lambda\phi^4$ Theory

Topics for Next Lecture:

13.1 Production of Higgs Boson from Gluon Fusion

12.5 Renormalization of the ϕ^4 theory

Renormalization is the procedure to remove ultraviolet (UV) divergences systematically in order to evaluate finite physical quantities.

There are two common regularization formalisms:

- Pauli-Villars regularization with an ultraviolet cutoff $\Lambda \to \infty$,

$$\frac{1}{p^2 - m^2 + \imath\epsilon} \to$$
$$\frac{1}{p^2 - m^2 + \imath\epsilon} - \frac{1}{p^2 - \Lambda^2 + \imath\epsilon} = \frac{m^2 - \Lambda^2}{(p^2 - m^2 + \imath\epsilon)(p^2 - \Lambda^2 + \imath\epsilon)} \, .$$

- Dimensional regularization with $N = 4 - 2\epsilon$.

Weinberg theorem for renormalizable theories:
for a renormalizable theory, we only need finite number of counter terms to renormalize the fields, the masses, and couplings, such that physical observables are finite to all orders in perturbation theory.

A. One loop structure of ϕ^4 theory

In $N-$dimensions the action of the ϕ^4 theory is

$$S = \int \mathcal{L}_{\rm B}\, d^N x \quad \text{with} \quad N = 4 - 2\epsilon$$

and the bare Lagrangian density is

$$\mathcal{L}_{\rm B} = \frac{1}{2}(\partial_\mu \phi_0)(\partial^\mu \phi_0) - \frac{1}{2} m_0^2 \phi_0^2 - \frac{1}{4!}\lambda_0^2 \phi_0^4$$

where

- $\phi_0 =$ the bare field with $[\phi_0] = 1 - \epsilon$,
- $m_0 =$ the bare mass with $[m_0] = 1$,
- $\lambda_0 =$ the bare coupling with $[\lambda_0] = 2\epsilon$.

We often define

$$\begin{aligned} \phi_0 &= \mu^{-\epsilon} Z_\phi^{1/2} \phi \\ m_0 &= Z_m^{1/2} m \\ \lambda_0 &= \mu^{2\epsilon} Z_\lambda \lambda \end{aligned}$$

where

- $\phi =$ the renormalized field with $[\phi] = 1$,
- $m =$ the renormalized mass with $[m] = 1$,
- $\lambda =$ the renormalized coupling with $[\lambda] = 0$.

Then the action becomes

$$\begin{aligned} S &= \mu^{-2\epsilon} \int \left[\mathcal{L}_{\mathrm{R}} + \mathcal{L}_{\mathrm{CT}}\right] d^N x \\ &= \mu^{-2\epsilon} \int \left[\frac{1}{2} Z_\phi \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} Z_m Z_\phi m^2 \phi^2 - \frac{1}{4!} (Z_\lambda Z_\phi^2) \lambda \phi^4\right] d^{4-2\epsilon} x\,. \end{aligned}$$

The renormalized Lagrangian density is

$$\begin{aligned}\mathcal{L}_{\mathrm{R}} &= \mathcal{L}_0 + \mathcal{L}_{\mathrm{I}} \\ &= \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!}\lambda \phi^4\end{aligned}$$

with the kinetic energy or the unperturbed Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

and the ordinary perturbation or the interaction Lagrangian

$$\mathcal{L}_{\mathrm{I}} = -\frac{1}{4!}\lambda^2 \phi^4 \,.$$

And the counter term Lagrangian is

$$\mathcal{L}_{\mathrm{CT}} = (Z_\phi - 1)\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - (Z_m Z\phi - 1)\frac{1}{2} m^2 \phi^2 - (Z_\lambda Z_\phi^2 - 1)\frac{1}{4!}\lambda^2 \phi^4 \,.$$

Now the complete set of Feynman rules become:

- propagator: $i/(p^2 - m^2 + i\epsilon)$,
- vertex: $-i\lambda$,
- counter term propagator: $i[(Z_\phi - 1)p^2 - (Z_m Z_\phi - 1)m^2$,
- counter term vertex: $-i\lambda(Z_\lambda Z_\phi^2 - 1)$.

There is a factor

$$\mu^{2\epsilon} \int \frac{d^N \ell}{(2\pi)^N}$$

for all momentum integration.

Let us consider the various Z's in the following form

$$Z = 1 + \sum_{n=1}^{\infty} Z^{(n)} \lambda^n \quad \text{or} \quad Z = 1 + \sum_{n=1}^{\infty} (\delta Z)^n \lambda^n$$

then choose the coefficient $Z^{(n)}$ to remove the infinities.

Example:

At the one-loop order,

$$Z_\lambda Z_\phi^2 = 1 + \left(\frac{3}{16\pi^2\epsilon}\right)\lambda + \mathcal{O}(\lambda^2)\,.$$

Then we can remove the pole at order λ^2. In the minimal subtraction scheme, we remove exactly the pole with $1/\epsilon$.

In Section 7.1 of Peskin and Schroeder's book, we learned that the exact two-point function has the following form

$$\int \langle\Omega|T\phi(x)\phi(0)|\Omega\rangle e^{ip\cdot x}\, d^4x \;=\; \frac{iZ}{p^2 - m^2} + (\text{regular terms at } p^2 = m^2)\,.$$

we can eliminate the awkward residue Z from this equation by rescaling the field

$$\phi_0 = Z^{1/2}\phi_{\mathrm{R}}\,.$$

That means, we may set $Z_\phi = Z$.

Then the bare Lagrangian and the counter term Lagrangian become

$$\begin{aligned} \mathcal{L}_{\rm B} &= \frac{1}{2} Z (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m_0^2 Z \phi^2 - \frac{1}{4!} \lambda_0^2 Z^2 \phi^4 \,, \\ \mathcal{L}_{\rm CT} &= \frac{1}{2} (\delta Z)(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} (\delta m^2) m^2 \phi^2 - \frac{1}{4!} (\delta \lambda) \phi^4 \,. \end{aligned}$$

The bare mass and bare coupling are defined as

$$m_0^2 Z = m^2 + \delta m^2 \quad \text{and} \quad \lambda_0 Z^2 = \lambda + \delta\lambda$$

where $Z = 1 + \delta Z$; δZ, δm^2 and $\delta\lambda$ are the divergent counter terms.

Now the complete set of Feynman rules become:

- propagator: $i/(p^2 - m^2 + i\epsilon)$,
- vertex: $-i\lambda$,
- counter term propagator: $i[(\delta Z)p^2 - \delta m^2]$,
- counter term vertex: $-i\delta\lambda$.

Renormalization Condition

A good definition of λ is that the value of λ is equal to the magnitude of the scattering amplitude at zero momentum. Thus we have the two defining relations

- the full propagator is

$$\frac{1}{p^2 - m^2} + (\text{terms regular at } p^2 = m^2)\,,$$

- the amputated vertex is $-i\lambda$ at $s = 4m^2, t = u = 0$.

These equations are called the renormalization conditions. The first equation actually contains two conditions, specifying the location of the pole and its residue.

B. Mass Renormalization in $\lambda\phi^4$ Theory

Suppose we have calculated the self-energy diagram

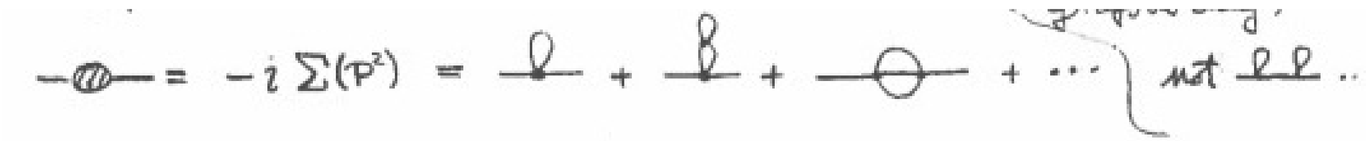

Figure 1: Self energy diagrams in $\lambda\phi^4$ theory.

The full propagator is

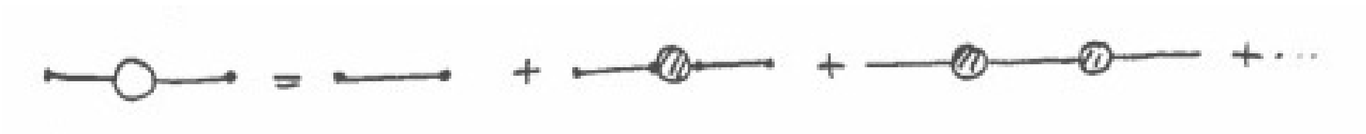

Figure 2: The full propagator of a spin-0 particle.

That is

$$
\begin{aligned}
& i\Delta(p^2) \\
& \quad = \frac{i}{p^2 - m^2 + i\epsilon} + \frac{i}{p^2 - m^2 + i\epsilon} \left(-i\Sigma(p^2)\right) \frac{i}{p^2 - m^2 + i\epsilon} \\
& \quad + \frac{i}{p^2 - m^2 + i\epsilon} \left(-i\Sigma(p^2)\right) \frac{i}{p^2 - m^2 + i\epsilon} \left(-i\Sigma(p^2)\right) \frac{i}{p^2 - m^2 + i\epsilon} + \cdots \\
& = \frac{i}{p^2 - m^2 + i\epsilon} + \frac{i}{p^2 - m^2 + i\epsilon} \left(-i\Sigma(p^2)\right) \left(i\Delta(p^2)\right)
\end{aligned}
$$

so

$$(p^2 - m^2)\Delta = 1 + \Sigma\Delta \quad \text{or} \quad (p^2 - m^2 - \Sigma)\Delta = 1$$

Therefore, the full propagator is

$$i\Delta(p^2) = \frac{i}{p^2 - m^2 - \Sigma}$$

In particular, if

$$\begin{aligned}
\Sigma_{\text{loop}}(p^2) &= \frac{\lambda}{32\pi^2} m^2 \Gamma(-1+\epsilon) \left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} \\
&= \frac{\lambda}{32\pi^2} m^2 [-\frac{1}{\epsilon} - 1 + \gamma + O(\epsilon)] \left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} \\
&= \frac{\lambda}{32\pi^2} (m^2) \left[-\frac{1}{\epsilon} + \gamma - \log(4\pi) - 1 + \log\left(\frac{m^2}{\mu^2}\right) + O(\epsilon)\right] .
\end{aligned}$$

The term with bare mass is regularized to contain a finite renormalized mas (m_R) and a divergent term (δm)

$$m_0^2 = m_R^2 + \delta m^2 \,.$$

The renormalized self-energy becomes

$$\Sigma_R(p, m_R) = \Sigma_{\text{loop}} + \Sigma_{\text{CT}} = \Sigma_{\text{loop}} + \delta m^2$$

with

$$\delta m^2 = \frac{\lambda}{32\pi^2}(m^2)\left(\frac{1}{\epsilon} - c_m\right) \; .$$

The physical mass squared then becomes

$$m^2_{\text{ph}} = m^2_R + \Sigma_R \; .$$

Renormalization Schemes

There are two simple renormalization schemes at the first order in λ.

(a) **The minimal subtraction scheme** (MS)

Let us choose $c_m = 0$. The renormalized self-energy becomes

$$\begin{aligned}\Sigma(p,m) &= \frac{\lambda}{32\pi^2}(m^2)\left[\gamma - \log(4\pi) - 1 + \log\left(\frac{m^2}{\mu^2}\right) + O(\epsilon)\right] \\ &= \frac{\lambda}{32\pi^2}(m^2)\log\left[\left(\frac{m^2}{(4\pi)\mu^2}\right)e^{\gamma-1}\right]\end{aligned}$$

and the physical mass is

$$M^2 = m_{\mathrm{ph}}^2 = \frac{\lambda}{32\pi^2}(m^2)\left[1 + \log\left(\frac{m^2}{(4\pi)\mu^2}e^{\gamma-1}\right)\right] .$$

(b) **The modified minimal subtraction scheme** ($\overline{\mathrm{MS}}$)

It was suggested by Bardeen et al. They found that the combination

$$\Delta_\epsilon = \frac{1}{\epsilon} - \gamma + \log(4\pi)$$

always appears in the dimensional regularization. Thus it is convenient to choose

$$c_m = \gamma - \log(4\pi) \; .$$

Then the renormalized self-energy becomes

$$\begin{aligned}\Sigma_R(p,m) &= \Sigma_{\mathrm{loop}} + \Sigma_{\mathrm{CT}} \\ &= \frac{\lambda}{32\pi^2}(m^2)\left[-1+\log\left(\frac{m^2}{\mu^2}\right)\right] \;=\; \frac{\lambda}{32\pi^2}(m^2)\log\left[\left(\frac{m^2}{\mu^2}\right)e^{-1}\right]\end{aligned}$$

and the physical mass is

$$m^2_{\mathrm{ph}} = \frac{\lambda}{32\pi^2}(m^2)\left[1+\log\left(\frac{m^2}{\mu^2}e^{-1}\right)\right] \; .$$

The on-shell scheme

A third scheme is called the momentum subtraction scheme or the on-shell scheme (OS).

Recall that the physical mass is the solution of

$$M^2 = m_{\rm ph}^2 = m_R^2 + \Sigma(p, m_R) \ .$$

Let us define

$$\Sigma'(p^2) \equiv \frac{d\Sigma}{dp^2} \ .$$

Then near the pole we can expand

$$\Sigma(p^2) \sim \Sigma(M^2) + (p^2 - M^2)\Sigma'(p^2)|_{p^2=M^2} \ .$$

That leads to

$$
\begin{aligned}
i\Delta(p^2) &\simeq \frac{i}{p^2 - m^2 - \Sigma(M^2) - (p^2 - M^2)\Sigma'(p^2)} \\
&= \frac{i}{(p^2 - M^2 + i\epsilon)[1 - \Sigma'(M^2)]}
\end{aligned}
$$

where we have chosen

$$M^2 = m_R^2 + \Sigma(M^2)\ .$$

In the on-shell renormalization scheme, we adjust the Z's so that

$$m = M_{\text{physical}} \ \text{ i.e. } \ \Sigma(m^2) = 0$$

and the residue of the pole is one

$$\Sigma'(m^2) = 0\ .$$

C. Renormalization of Coupling in $\lambda\phi^4$ Theory

Recall that at the one-loop level the 4-point Green function is

$$\begin{aligned} G_4^{(1)} &= i(M_1 + M_2 + M3) \\ &= i\frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[3\left(\frac{1}{\epsilon} - \gamma + \log(4\pi)\right) + F_4\right] + \mathcal{O}(\epsilon) \\ &= i\frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[3\Delta_\epsilon + F_4\right] + \mathcal{O}(\epsilon)\,. \end{aligned}$$

To the order of the λ^2, the 4-point Green function becomes

$$\begin{aligned}
G_4 &= G_4^{(0)} + G_4^{(1)} \; = \; i\mathcal{M} \\
&= -i\lambda + i\frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[3\Delta_\epsilon + F_4\right] + \mathcal{O}(\epsilon) \quad \text{with} \\
\Delta_\epsilon &= \frac{1}{\epsilon} - \gamma + \log(4\pi)\,, \quad \text{and} \\
F_4 &= +3\log\left(\frac{\mu^2}{m^2}\right) - \int_0^1 dx\, \log\left(1 + \frac{-sx(1-x)}{m^2}\right) \\
&\quad - \int_0^1 dx\, \log\left(1 + \frac{-tx(1-x)}{m^2}\right) - \int_0^1 dx\, \log\left(1 + \frac{-ux(1-x)}{m^2}\right).
\end{aligned}$$

Integrating over x, we obtain

$$
\begin{aligned}
F_4 \quad = \quad & 6 + 3\log\left(\frac{\mu^2}{m^2}\right) - \sqrt{1 - \frac{4m^2}{s}} \log \frac{\sqrt{1-\frac{4m^2}{s}} + 1}{\sqrt{1-\frac{4m^2}{s}} - 1} \\
& - \sqrt{1 - \frac{4m^2}{t}} \log \frac{\sqrt{1-\frac{4m^2}{t}} + 1}{\sqrt{1-\frac{4m^2}{t}} - 1} - \sqrt{1 - \frac{4m^2}{u}} \log \frac{\sqrt{1-\frac{4m^2}{u}} + 1}{\sqrt{1-\frac{4m^2}{u}} - 1} \,.
\end{aligned}
$$

Then the renormalized vertex function becomes

$$
iM = -i\lambda + (-i\lambda)^2 [iV(s) + iV(t) + iV(u)] - i\delta\lambda \,.
$$

Applying the renormalization condition for the vertex function, we need this amplitude to be equal to $-i\lambda$ at zero momentum with $s = 4m^2$ and $t = u = 0$. Therefore, we must set

$$
\delta\lambda = -\lambda^2 [V(4m^2) + 2V(0)]
$$

That determines the counter term of the coupling constant in the on-shell scheme as

$$\delta\lambda = \frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[3\Delta_\epsilon + 3\log\left(\frac{\mu^2}{m^2}\right) - \log\left(1-4x(1-x)\right)\right].$$

Then the finite result becomes

$$\begin{aligned} i\mathcal{M} &= -i\lambda - i\frac{\lambda^2}{32\pi^2}\int_0^1 dx\, F(s,t,u) \\ F(s,t,u) &= \log\left(\frac{-x(1-x)s+m^2}{-x(1-x)(4m^2)+m^2}\right) \\ &\quad + \log\left(\frac{-x(1-x)t+m^2}{m^2}\right) \\ &\quad + \log\left(\frac{-x(1-x)u+m^2}{m^2}\right) \end{aligned}$$

with on-shell renormalization.