PHYS 6213: Advanced Particle Physics, Spring 2022

Lecture 24, Apr 13, 2022 (Wednesday)

- Reading:
  (a) Chap 12 in Collider Physics
  (b) Chap 18 and Chap 19 in Quantum Field Theory

- Assignments:
  (a) Term Paper 1: $pp \to H + X$ due Apr 28 (Thu)
  (b) Term Paper 2: $pp \to VH + X$ due May 06 (Fri)

Topics for Today:

Chapter 12 Dimensional Regularization and Renormalization

12.1 Loop Integrals in N Dimensions

12.2 Dimensional Regularization

12.3 Self-Energy in $\lambda\phi^4$ Theory

12.4 Vertex Corrections in $\lambda\phi^4$ Theory

Topics for Next Lecture:

12.5 Renormalization in $\lambda\phi^4$ Theory

12.1 Loop Integrals in N Dimensions

Applying generalized spherical coordinates in N-dimensions, we have found

$$\int \frac{d^N \ell}{(\ell^2 + M^2)^A} = \pi^{N/2} \frac{\Gamma(A - N/2)}{\Gamma(A)} \times \frac{1}{(M^2)^{A-N/2}} \,. \tag{1}$$

Now, by letting $q = \ell + p$ we can write Eq. (1) in the following form,

$$\int \frac{d^N \ell}{(\ell^2 + 2\ell \cdot p + M^2)^A} = \pi^{N/2} \frac{\Gamma(A - N/2)}{\Gamma(A)} \frac{1}{(M^2 - p^2)^{A-N/2}} \,.$$

Next by successive differentiation of previous equation with respect to p_μ, it is easy to obtain the formula

$$\int d^N\ell \frac{\ell_\mu}{(\ell^2 + 2\ell\cdot p + M^2)^A} = \pi^{N/2}\frac{\Gamma(A - N/2)}{\Gamma(A)}\frac{(-2p_\mu)}{(M^2 - p^2)^{A-N/2}}$$

and

$$\begin{aligned} & \int d^N\ell \, \frac{\ell_\mu \ell_\nu}{(\ell^2 + 2\ell\cdot p + M^2)^A} \\ = \; & \frac{\pi^{N/2}}{\Gamma(A)(M^2 - p^2)^{A-N/2}} \\ & \times \left[\Gamma(A - N/2)p_\mu p_\nu + \frac{1}{2}\delta_{\mu\nu}\Gamma(A - 1 - N/2)(M^2 - p^2)\right] \, . \end{aligned}$$

For N = 4, the integral I_N becomes

$$I = \pi^2 \int_0^\infty d\ell^2 \ell^2 F(\ell^2) \, . \qquad (2)$$

The Gamma Functions

The following formulas are very useful for dimensional regularization.

$$\begin{aligned} \Gamma(-n+\epsilon) &= \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon}+\psi_1(n+1)+O(\epsilon)\right] \\ \Gamma(\epsilon) &= \frac{1}{\epsilon}-\gamma+O(\epsilon) \\ \Gamma(-1+\epsilon) &= -\left[\frac{1}{\epsilon}+1-\gamma+O(\epsilon)\right] \\ \Gamma(-2+\epsilon) &= \frac{1}{2}\left[\frac{1}{\epsilon}+1+\frac{1}{2}-\gamma+O(\epsilon)\right] \end{aligned} \tag{3}$$

where γ is the Euler constant, and

$$\begin{aligned} \psi_1(z) &= \frac{d[\ln\Gamma(z)]}{dz} = \frac{\Gamma'(z)}{\Gamma(z)} \\ \psi_1(n+1) &= 1+\frac{1}{2}+\ldots+\frac{1}{n}-\gamma \end{aligned} \tag{4}$$

Feynman Parametrization for Loop Integrals

$$\frac{1}{d_1 d_2} = \Gamma(2) \int_0^1 dx \frac{1}{[x(d_2 - d_1) + d_1]^2}$$

$$\frac{1}{d_1 d_2 d_3} = \Gamma(3) \int_0^1 dx \int_0^x dy \frac{1}{[x(d_2 - d_1) + y(d_3 - d_2) + d_1]^3}$$

$$= \Gamma(3) \int_0^1 dx \int_0^x dy \frac{1}{[x(d_2 - d_3) + y(d_1 - d_2) + d_3]^3}$$

$$\frac{1}{d_1 d_2 d_3 d_4}$$

$$= \int_0^1 dx \int_0^x dy \int_0^y dz \frac{\Gamma(4)}{[x(d_2 - d_1) + y(d_3 - d_2) + z(d_4 - d_3) + d_1]^4}$$

$$= \int_0^1 dx \int_0^x dy \int_0^y dz \frac{\Gamma(4)}{[x(d_3 - d_4) + y(d_2 - d_3) + z(d_1 - d_2) + d_4]^4} .$$

12.2 Dimensional Regularization

In high energy theory, there are 3 types of divergences: (a) ultraviolet ($E \to \infty$), (b) infrared (($E \to \epsilon \to 0+$), and (c) collinear divergence ($\cos\theta \to \pm 1$) between a quark and a gluon:

(a) ultraviolet divergence can be removed by renormalization with a high energy cut off or dimensional regularization;

(b) infrared divergence can be removed with real gluon or photon emission and a low energy cut off or dimensional regularization;

(c) collinear divergence can be removed by redefinition of parton distribution functions.

Regularization is the procedure to isolate divergences and determine the finite part for physical observables.

Example 1: For mass renormalization, we have

$$\Gamma(-1+\epsilon) = -\left[\frac{1}{\epsilon} + 1 - \gamma + O(\epsilon)\right] .$$

Example 2: For coupling renormalization, we have

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon) .$$

12.3 Self-Energy in $\lambda\phi^4$ Theory

In N dimensions with $N = 4 - 2\epsilon$, the Lagrangian density for the ϕ^4 theory becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\mu^{2\epsilon}\phi^4 \,.$$

Applying Feynman rules, we obtain the self energy as a one-loop integral

$$\begin{aligned} -i\Sigma(m^2) &= S(-i\lambda)\mu^{2\epsilon}\int \frac{d^N\ell}{(2\pi)^N}\frac{i}{\ell^2 - m^2 + i\epsilon} \\ &= \frac{1}{2}(-i\lambda)\mu^{2\epsilon}\int \frac{d^N\ell}{(2\pi)^N}\frac{i}{\ell^2 - m^2 + i\epsilon} \end{aligned}$$

where S is the symmetry factor

$$S = \frac{4\cdot 3}{4!} = \frac{1}{2}\,.$$

That leads to

$$\Sigma(m^2) = \frac{1}{2}(i\lambda)\mu^{2\epsilon} \int \frac{d^N\ell}{(2\pi)^N} \frac{1}{\ell^2 - m^2 + i\epsilon} \, .$$

Introducing Wick rotation with $\ell^0_M = i\ell^N_E$ and $\ell^2_M = -\ell^2_E = -\ell^2$, we obtain

$$\begin{aligned} \Sigma(m^2) &= \frac{1}{2}(i\lambda)\mu^{2\epsilon}(2\pi)^{-N} \int i d^N\ell_E \, \frac{-1}{\ell^2_E + m^2 - i\epsilon} \\ &= \frac{\lambda}{2}\mu^{2\epsilon}(2\pi)^{-N} \int d^N\ell \, \frac{1}{\ell^2 + m^2 - i\epsilon} \, . \end{aligned}$$

This is similar to the following integral in the N-dimensional Euclidean space,

$$\int d^N\ell \, \frac{1}{(\ell^2 + M^2)^A} = \pi^{N/2} \frac{\Gamma(A - N/2)}{\Gamma(A)} \frac{1}{(M^2)^{A-N/2}}$$

with $A=1$ and $M^2 = m^2 - i\epsilon$. Thus the self energy becomes

$$\Sigma(m^2) = \frac{\lambda}{2}\mu^{2\epsilon}(2\pi)^{-N}\pi^{N/2}\frac{\Gamma(1-N/2)}{\Gamma(1)}\frac{1}{(m^2-i\epsilon)^{1-N/2}}\,.$$

Choosing $N = 4-2\epsilon$, we obtain

$$\begin{aligned}\Sigma_{\rm loop}(m^2) &= \frac{\lambda}{32\pi^2}m^2\Gamma(-1+\epsilon)\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon}\\ &= \frac{\lambda}{32\pi^2}m^2\left[-\frac{1}{\epsilon}-1+\gamma+O(\epsilon)\right]\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon}\\ &= \frac{\lambda}{32\pi^2}(m^2)\left[-\Delta_\epsilon-1+\log\left(\frac{m^2}{\mu^2}\right)+O(\epsilon)\right]\,.\end{aligned}$$

where

$$\Delta_\epsilon = \frac{1}{\epsilon}-\gamma+\ln(4\pi)\,.$$

We have applied

$$\frac{\pi^{N/2}}{(2\pi)^N} = \frac{\pi^{N/2}}{(4\pi^2)^{N/2}} = (4\pi)^{-(N/2)} = (4\pi)^{-(2-\epsilon)} = \frac{1}{16\pi^2}(4\pi)^{\epsilon},$$

as well as

$$\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} = e^{\ln\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon}} = e^{-\epsilon\ln\left(\frac{m^2}{4\pi\mu^2}\right)} = 1 - \epsilon\ln\left(\frac{m^2}{4\pi\mu^2}\right) + \mathcal{O}(\epsilon^2),$$

and

$$\Gamma(-1+\epsilon) = (-1)\left[\frac{1}{\epsilon} + 1 - \gamma + \mathcal{O}(\epsilon)\right]$$

where γ is the Euler constant and ϵ is an infinitesimal parameter.

12.4 Vertex Corrections in $\lambda\phi^4$ Theory

In N dimensions with $N = 4 - 2\epsilon$, we need to define a new coupling

$$\lambda_{\rm old} = \lambda_{\rm new} \cdot \mu^{2\epsilon} = \lambda \mu^{2\epsilon}$$

$\lambda = \lambda_{\rm new}$ to keep λ dimensionless so this theory is renormalizable. Then the Lagrangian density becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\mu^{2\epsilon}\phi^4 \,.$$

The Next-to-Leading-Order (NLO) vertex corrections have contributions from s, t and $u-$channel diagrams. Applying Feynman rules, we obtain the transition amplitude for the $s-$channel contribution as

$$
\begin{aligned}
iM_1 &= \frac{1}{2}(-i\lambda)^2(\mu^{2\epsilon})^2 \int \frac{d^N\ell}{(2\pi)^N} \frac{i}{\ell^2 - m^2 + i\epsilon} \frac{i}{(\ell + q_1)^2 - m^2 + i\epsilon} \\
&= \frac{1}{2}\lambda^2(\mu^2)^{2\epsilon}(2\pi)^{-N} \int d^N\ell \, \frac{1}{(\ell^2 - m^2 + i\epsilon)[(\ell + q_1)^2 - m^2 + i\epsilon]} \\
&= \frac{1}{2}\lambda^2(\mu^2)^{2\epsilon}(2\pi)^{-N} \int d^N\ell \, \frac{1}{d_1 d_2}
\end{aligned}
$$

where

$$
\begin{aligned}
d_1 &= \ell^2 - m^2 + i\epsilon \\
d_2 &= (\ell + q_1)^2 - m^2 + i\epsilon
\end{aligned}
$$

with $q_1 = p_1 + p_2$ and $q_1^2 = s$.

Introducing a Feynman parameter x, we obtain

$$\frac{1}{d_1 d_2} = \int_0^1 dx \, \frac{1}{[x(d_2 - d_1) + d_1]^2}$$
$$= \int_0^1 dx \, \frac{1}{[\ell^2 + 2\ell \cdot q_1 x - m^2 + x q_1^2)^2}$$

where

$$d_2 - d_1 = 2\ell \cdot q_1 + q_1^2 \, .$$

Then the amplitude becomes

$$iM_1 = \frac{1}{2}(\lambda)^2 (\mu^2)^{2\epsilon} (2\pi)^{-N} \int d^N\ell \, \frac{1}{d_1 d_2}$$
$$= \frac{1}{2}(\lambda)^2 (\mu^2)^{2\epsilon} (2\pi)^{-N} \int d^N\ell \int_0^1 dx \, \frac{1}{[\ell^2 + 2\ell \cdot q_1 x - m^2 + x q_1^2)^2} \, .$$

In N dimensions with dimensional regularization, we can switch the order of integration and express M_1 as

$$iM_1 = \frac{1}{2}(\lambda)^2(\mu^2)^{2\epsilon}(2\pi)^{-N}\int_0^1 dx \int d^N\ell \, \frac{1}{[\ell^2 + 2\ell\cdot q_1 x - m^2 + xq_1^2)^2} \, .$$

The integral in M_1 can be simplified with a shift $q = \ell + xq_1$ or $\ell = q - xq_1$. Then the amplitude becomes

$$iM_1 = \frac{1}{2}(\lambda)^2(\mu^2)^{2\epsilon}(2\pi)^{-N}\int_0^1 dx \int d^N q \, \frac{1}{[q^2 + x(1-x)q_1^2 - m^2]^2} \, .$$

Introducing Wick rotation in the complex ℓ^0 plane with $\ell^0 = i\ell_E^N$ and $\ell_E^N \in \mathcal{R}$, we obtain

$$d^N\ell_M = i d^N\ell_E \quad \text{and} \quad \ell_M^2 = (\ell^0)^2 - |\vec{\ell}|^2 = -(\ell^N)^2 - |\vec{\ell}|^2 = -\ell_E^2 = -\ell^2 \, .$$

Then the s-channel amplitude becomes

$$iM_1 = \\ i(-1)^2 \frac{1}{2}(\lambda)^2(\mu^2)^{2\epsilon}(2\pi)^{-N} \int_0^1 dx \int d^N q_E \, \frac{1}{[q^2 - x(1-x)q_1^2 + m^2]^2} \, .$$

Recall that in the $N-$dimensional Euclidean space, we have

$$I_N = \int d^N \ell \, \frac{1}{(\ell^2 + M^2)^A} = \pi^{N/2} \frac{\Gamma(A - N/2)}{\Gamma(A)} \frac{1}{(M^2)^{A-N/2}} \, .$$

Thus we now have

$$\begin{aligned} \int d^N q \, \frac{1}{[q^2 - x(1-x)q_1^2 + m^2]^2} &= \int d^N q \, \frac{1}{[q^2 + M^2]^2} \\ &= \pi^{N/2} \frac{\Gamma(2 - N/2)}{\Gamma(2)} \frac{1}{(M^2)^{2-N/2}} \end{aligned}$$

with $A = 2$ and $M^2 = -x(1-x)q_1^2 + m^2$.

The s-channel diagram becomes

$$M_1 =$$
$$\frac{1}{2}(\lambda)^2(\mu^2)^{2\epsilon}(2\pi)^{-N}\pi^{N/2}\Gamma(2-N/2)\int_0^1 dx\,\frac{1}{[-x(1-x)q_1^2+m^2]^{2-N/2}}$$

with $\Gamma(2) = 1! = 1$.

We often choose $N = 4-2\epsilon$. Then $2-N/2 = \epsilon$, $\Gamma(2-N/2) = \Gamma(\epsilon)$, and

$$\frac{\pi^{N/2}}{(2\pi)^N} = \frac{\pi^{N/2}}{(4\pi^2)^{N/2}} = (4\pi)^{-(N/2)} = (4\pi)^{-(2-\epsilon)} = \frac{1}{16\pi^2}(4\pi)^{\epsilon}\,.$$

Then the amplitude becomes

$$\begin{aligned} M_1 &= \frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}(\mu^2)^{\epsilon}(4\pi)^{\epsilon})\Gamma(\epsilon)\int_0^1 dx\,[m^2+x(1-x)q_1^2]^{-\epsilon} \\ &= \frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\Gamma(\epsilon)\int_0^1 dx\,\left[\frac{-x(1-x)q_1^2+m^2}{4\pi\mu^2}\right]^{-\epsilon}. \end{aligned}$$

Now let us consider

$$\Lambda^{-\epsilon} = e^{\ln\left(\Lambda^{-\epsilon}\right)} = e^{-\epsilon \ln(\Lambda)} = 1 - \epsilon \ln(\Lambda) + \mathcal{O}(\epsilon^2)$$

and

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!} \left[\frac{1}{\epsilon} + \psi(n+1) + \mathcal{O}(\epsilon)\right]$$

where

$$\begin{aligned} \psi(n+1) &= 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma\,, \\ \psi(1) &= -\gamma \simeq -0.5772\,, \end{aligned}$$

γ is the Euler constant and $\epsilon \to 0+$ is an infinitesimal positive parameter.

Then the amplitude becomes

$$M_1 = \frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[\frac{1}{\epsilon} - \gamma - \int_0^1 dx \ \ln\left(\frac{m^2 - x(1-x)q_1^2}{4\pi\mu^2}\right) + \mathcal{O}(\epsilon)\right]$$

$$= \frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[\frac{1}{\epsilon} - \gamma + \ln(4\pi) + \ln\left(\frac{\mu^2}{m^2}\right) - \int_0^1 dx \ \ln\left(1 + \frac{-x(1-x)q_1^2}{m^2}\right) + \right.$$

$$= \frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[\Delta_\epsilon + \ln\left(\frac{\mu^2}{m^2}\right) - \int_0^1 dx \ \ln\left(1 + \frac{-q_1^2}{4m^2}4x(1-x)\right) + \mathcal{O}(\epsilon)\right]$$

where

$$\Delta_\epsilon = \frac{1}{\epsilon} - \gamma + \ln(4\pi)$$

and

$$\ln\left(\frac{m^2 - x(1-x)q_1^2}{4\pi\mu^2}\right) = -\ln(4\pi) - ln\left(\frac{\mu^2}{m^2}\right) + \ln\left(1 + \frac{-q_1^2}{4m^2}4x(1-x)\right) .$$

Applying the following integral

$$\int_0^1 dx\, \left[1 + \frac{4}{a} x(1-x)\right] = -2 + \sqrt{1+a}\,\ln\left(\frac{\sqrt{1+a}+1}{\sqrt{1+a}-1}\right) \quad \text{for} \quad a > 0$$

we obtain

$$\int_0^1 dx\, \ln\left(\frac{m^2 - x(1-x)q_1^2}{4\pi\mu^2}\right) =$$

$$-\ln(4\pi) - ln\left(\frac{\mu^2}{m^2}\right) + \left[-2 + \sqrt{1 - \frac{4m^2}{q_1^2}}\,\ln\frac{\sqrt{1-\frac{4m^2}{q_1^2}}+1}{\sqrt{1-\frac{4m^2}{q_1^2}}-1}\right].$$

Then the s-channel diagram with $q_1^2 = s$ becomes

$$M_1 = \frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[\frac{1}{\epsilon} - \gamma + \ln(4\pi)\right.$$

$$\left. +2 + ln\left(\frac{\mu^2}{m^2}\right) - \sqrt{1 - \frac{4m^2}{s}}\,\ln\frac{\sqrt{1-\frac{4m^2}{s}}+1}{\sqrt{1-\frac{4m^2}{s}}-1}\right] + \mathcal{O}(\epsilon)\,.$$

Similarly, the $t-$ and $u-$channel diagrams become

$$M_2 = \frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[\Delta_\epsilon + 2 + ln\left(\frac{\mu^2}{m^2}\right) - \sqrt{1-\frac{4m^2}{t}}\ \mathrm{ln}\,\frac{\sqrt{1-\frac{4m^2}{t}}+1}{\sqrt{1-\frac{4m^2}{t}}-1}\right] + \mathcal{O}(\epsilon)\,,$$

and

$$M_3 = \frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[\Delta_\epsilon + 2 + ln\left(\frac{\mu^2}{m^2}\right) - \sqrt{1-\frac{4m^2}{u}}\ \mathrm{ln}\,\frac{\sqrt{1-\frac{4m^2}{u}}+1}{\sqrt{1-\frac{4m^2}{u}}-1}\right] + \mathcal{O}(\epsilon)$$

with $q_2^2 = (p_2 + p3)^2 = t$ and $q_3^2 = (p_1 + p3)^2 = u$.

Thus the total one-loop contribution at the order of λ^2 is

$$\begin{aligned} G_4^{(1)} &= i(M_1 + M_2 + M3) \\ &= i\frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[3\Delta_\epsilon + F_4\right] + \mathcal{O}(\epsilon)\,. \end{aligned}$$

To the order of the λ^2, the 4-point vertex function or the 4-point Green function becomes

$$
\begin{aligned}
G_4 &= G_4^{(0)} + G_4^{(1)} = -i\lambda + i\frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[3\Delta_\epsilon + F_4\right] + \mathcal{O}(\epsilon) \quad \text{with} \\
\Delta_\epsilon &= \frac{1}{\epsilon} - \gamma + \ln(4\pi)\,, \quad \text{and} \\
F_4 &= 6 + 3ln\left(\frac{\mu^2}{m^2}\right) - \sqrt{1-\frac{4m^2}{s}}\ln\frac{\sqrt{1-\frac{4m^2}{s}}+1}{\sqrt{1-\frac{4m^2}{s}}-1} \\
&\quad -\sqrt{1-\frac{4m^2}{t}}\ln\frac{\sqrt{1-\frac{4m^2}{t}}+1}{\sqrt{1-\frac{4m^2}{t}}-1} - \sqrt{1-\frac{4m^2}{u}}\ln\frac{\sqrt{1-\frac{4m^2}{u}}+1}{\sqrt{1-\frac{4m^2}{u}}-1}\,.
\end{aligned}
$$

The divergent part

$$
+i\frac{3\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[\frac{1}{\epsilon}\right]
$$

needs to be removed by renormalization with counter terms.

12.5 Renormalization of the ϕ^4 theory

Renormalization is the procedure to remove ultraviolet (UV) divergences systematically in order to evaluate finite physical quantities.

A. One loop structure of ϕ^4 theory

In $N-$dimensions the action of the ϕ^4 theory is

$$S = \int d^N x \, \mathcal{L}_\mathrm{B} \quad \text{with} \quad N = 4 - 2\epsilon$$

and the bare Lagrangian density is

$$\mathcal{L}_\mathrm{B} = \frac{1}{2}(\partial_\mu \phi_0)(\partial^\mu \phi_0) - \frac{1}{2} m_0^2 \phi_0^2 - \frac{1}{4!} \lambda_0^2 \phi_0^4$$

where

- $\phi_0 =$ the bare field with $[\phi_0] = 1 - \epsilon$,

- $m_0 =$ the bare mass with $[m_0] = 1$,
- $\lambda_0 =$ the bare coupling with $[\lambda_0] = 2\epsilon$.

We often define

$$
\begin{aligned}
\phi_0 &= \mu^{-\epsilon} Z_\phi^{1/2} \phi \\
m_0 &= Z_m^{1/2} m \\
\lambda_0 &= \mu^{2\epsilon} Z_\lambda \lambda
\end{aligned}
$$

where

- $\phi =$ the renormalized field with $[\phi] = 1$,
- $m =$ the renormalized mass with $[m] = 1$,
- $\lambda =$ the renormalized coupling with $[\lambda] = 0$.

Then the action becomes

$$
\begin{aligned}
S &= \mu^{-2\epsilon} \int d^{4-2\epsilon}x \left[(Z_\phi) \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - (Z_m Z \phi) \frac{1}{2} m^2 \phi^2 - (Z_\lambda Z_\phi^2) \frac{1}{4!} \lambda \phi^4 \right] \\
&= \mu^{-2\epsilon} \int d^{4-2\epsilon}x \left[\mathcal{L}_{\mathrm{R}} + \mathcal{L}_{\mathrm{CT}} \right]
\end{aligned}
$$

where $\mathcal{L}_{\mathrm{R}}$ is the renormalized Lagrangian density

$$
\begin{aligned}
\mathcal{L}_{\mathrm{R}} &= \mathcal{L}_0 + \mathcal{L}_{\mathrm{I}} \\
&= \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4
\end{aligned}
$$

with the kinetic energy or the unperturbed Lagrangian

$$
\mathcal{L}_0 = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2
$$

and the ordinary perturbation or the interaction Lagrangian

$$
\mathcal{L}_{\mathrm{I}} = -\frac{1}{4!} \lambda^2 \phi^4 \, .
$$

And the counter term Lagrangian is

$$\mathcal{L}_{\rm CT} = (Z_\phi - 1)\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - (Z_m Z\phi - 1)\frac{1}{2}m^2\phi^2 - (Z_\lambda Z_\phi^2 - 1)\frac{1}{4!}\lambda^2\phi^4 \,.$$

Now the complete set of Feynman rules become:

- propagator: $i/(p^2 - m^2 + i\epsilon)$,
- vertex: $-i\lambda$,
- counter term propagator: $i[(Z_\phi - 1)p^2 - (Z_m Z_\phi - 1)m^2$,
- counter term vertex: $-i\lambda(Z_\lambda Z_\phi^2 - 1)$.

There is a factor

$$\mu^{2\epsilon}\int \frac{d^N\ell}{(2\pi)^N}$$

for all momentum integration.

Let us consider the various Z's in the following form

$$Z = 1 + \sum_{n=1}^{\infty} Z^{(n)} \lambda^n$$

then choose the coefficient $Z^{(n)}$ to remove the infinities.

Example:

At the one-loop order,

$$Z_\lambda Z_\phi^2 = 1 + \left(\frac{3}{16\pi^2 \epsilon} \right) \lambda + \mathcal{O}(\lambda^2) \, .$$

Then we can remove the pole order λ^2. In the minimal subtraction scheme, we remove exactly the pole with $1/\epsilon$.

In Section 7.1 of Peskin and Schroeder's book, we learned that the exact two-point function has the following form

$$\int d^4x \, \langle \Omega | T \phi(x) \phi(0) | \Omega \rangle e^{ip\cdot x} = \frac{iZ}{p^2 - m^2} + (\text{regular terms at} p^2 = m^2) \, .$$

we can eliminate the awkward residue Z from this equation by rescaling the field

$$\phi_0 = Z^{1/2}\phi_{\rm R}\,.$$

That means, we may set $Z_\phi = Z$.

Then the bare Lagrangian becomes

$$\mathcal{L}_{\rm B} = \int d^4x\,\left[\frac{1}{2}Z(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m_0^2 Z\phi^2 - \frac{1}{4!}\lambda_0^2 Z^2\phi^4\right]\,.$$

The bare mass and bare coupling can be defined as

$$\begin{aligned} Z &= 1 + \delta Z \\ m_0^2 Z &= m^2 + \delta m^2 \\ \lambda_0 Z^2 &= \lambda + \delta\lambda \end{aligned}$$

where δZ, δm^2 and $\delta\lambda$ are the divergent counter terms.

The counter term Lagrangian becomes

$$\mathcal{L}_{\rm CT} = \frac{1}{2}(\delta Z)(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}(\delta m^2) m^2 \phi^2 - \frac{1}{4!}(\delta\lambda)\phi^4 \,.$$

Now the complete set of Feynman rules become:

- propagator: $i/(p^2 - m^2 + i\epsilon)$,
- vertex: $-i\lambda$,
- counter term propagator: $i[(\delta Z)p^2 - \delta m^2]$,
- counter term vertex: $-i\delta\lambda$.

A good definition of λ is that the value of λ is equal to the magnitude of the scattering amplitude at zero momentum. Thus we have the two defining relations

- the full propagator is

$$\frac{1}{p^2 - m^2} + (\mathrm{terms\,regular\,at} p^2 = m^2) \,,$$

- the amputated vertex is $-i\lambda$ at $s = 4m^2, t = u = 0$.

These equations are called the renormalization conditions. The first equation actually contains two conditions, specifying the location of the pole and its residue.

B. Mass Renormalization in $\lambda\phi^4$ Theory

Suppose we have calculated the self-energy diagrams

$$-i\Sigma(p^2) =$$

The full propagator is

$$
\begin{aligned}
i\Delta(p^2) &= \frac{i}{p^2 - m^2 + i\epsilon} + \frac{i}{p^2 - m^2 + i\epsilon}\left(-i\Sigma(p^2)\right)\frac{i}{p^2 - m^2 + i\epsilon} \\
&\quad + \frac{i}{p^2 - m^2 + i\epsilon}\left(-i\Sigma(p^2)\right)\frac{i}{p^2 - m^2 + i\epsilon}\left(-i\Sigma(p^2)\right)\frac{i}{p^2 - m^2 + i\epsilon} + \cdots \\
&= \frac{i}{p^2 - m^2 + i\epsilon} + \frac{i}{p^2 - m^2 + i\epsilon}\left(-i\Sigma(p^2)\right)\left(i\Delta(p^2)\right)
\end{aligned}
$$

so

$$(p^2 - m^2)\Delta = 1 + \Sigma\Delta$$

that is

$$(p^2 - m^2 - \Sigma)\Delta = 1$$

Therefore, the full propagator is

$$i\Delta(p^2) = \frac{i}{p^2 - m^2 - \Sigma}$$

In particular, if

$$
\begin{aligned}
\Sigma_{\text{loop}}(p^2) &= \frac{\lambda}{32\pi^2} m^2 \Gamma(-1+\epsilon) \left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} \\
&= \frac{\lambda}{32\pi^2} m^2 [-\frac{1}{\epsilon} - 1 + \gamma + O(\epsilon)] \left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} \\
&= \frac{\lambda}{32\pi^2} (m^2) \left[-\frac{1}{\epsilon} + \gamma - \log(4\pi) - 1 + \log\left(\frac{m^2}{\mu^2}\right) + O(\epsilon)\right] .
\end{aligned}
$$

The term with bare mass is regularized to contain a finite renormalized mas (m_R) and a divergent term (δm)

$$m_0^2 = m_R^2 + \delta m^2$$

such that the renormalized self-energy is

$$\Sigma_R(p, m_R) = \Sigma_{\text{loop}} + \Sigma_{\text{CT}} = \Sigma_{\text{loop}} + \delta m^2$$

with

$$\delta m^2 = \frac{\lambda}{32\pi^2}(m^2)\left(\frac{1}{\epsilon} - c_m\right) \; .$$

The physical mass squared then becomes

$$m_{\text{ph}}^2 = m_R^2 + \Sigma_R \; .$$

Let us consider two simple renormalization schemes at the first order in λ.

(a) In **the minimal subtraction scheme** (MS), we choose $c_m = 0$. The renormalized self-energy becomes

$$\begin{aligned}
\Sigma(p,m) &= \frac{\lambda}{32\pi^2}(m^2)\left[\gamma - \log(4\pi) - 1 + \log\left(\frac{m^2}{\mu^2}\right) + O(\epsilon)\right] \\
&= \frac{\lambda}{32\pi^2}(m^2)\log\left[\left(\frac{m^2}{(4\pi)\mu^2}\right)e^{\gamma-1}\right]
\end{aligned}$$

and the physical mass is

$$M^2 = m_{\rm ph}^2 = \frac{\lambda}{32\pi^2}(m^2)\left[1 + \log\left(\frac{m^2}{(4\pi)\mu^2}e^{\gamma-1}\right)\right] .$$

(b) **The modified minimal subtraction scheme** ($\overline{\rm MS}$) was suggested by Bardeen et al. They found that the combination

$$\Delta_\epsilon = \frac{1}{\epsilon} - \gamma + \log(4\pi)$$

always appears in the dimensional regularization. Thus it is convenient to choose

$$c_m = \gamma - \log(4\pi) .$$

Then the renormalized self-energy becomes

$$\begin{aligned}\Sigma_R(p,m) &= \Sigma_{\text{loop}} + \Sigma_{\text{CT}} \\ &= \frac{\lambda}{32\pi^2}(m^2)\left[-1+\log\left(\frac{m^2}{\mu^2}\right)\right] \\ &= \frac{\lambda}{32\pi^2}(m^2)\log\left[\left(\frac{m^2}{\mu^2}\right)e^{-1}\right]\end{aligned}$$

and the physical mass is

$$m_{\text{ph}}^2 = \frac{\lambda}{32\pi^2}(m^2)\left[1+\log\left(\frac{m^2}{\mu^2}e^{-1}\right)\right] .$$

The on-shell scheme

A third scheme is called the momentum subtraction scheme or the on-shell scheme (OS).

Recall that the physical mass is the solution of

$$M^2 = m_{\text{ph}}^2 = m_R^2 + \Sigma(p, m_R) .$$

Let us define

$$\Sigma'(p^2) \equiv \frac{d\Sigma}{dp^2} \ .$$

Then near the pole we can expand

$$\Sigma(p^2) \sim \Sigma(M^2) + (p^2 - M^2)\Sigma'(p^2)|_{p^2=M^2}$$

so

$$\begin{aligned} i\Delta(p^2) &\simeq \frac{i}{p^2 - m^2 - \Sigma(M^2) - (p^2 - M^2)\Sigma'(p^2)} \\ &= \frac{i}{(p^2 - M^2 + i\epsilon)[1 - \Sigma'(M^2)]} \end{aligned}$$

where we have chosen

$$M^2 = m_R^2 + \Sigma(M^2) \ .$$

In the on-shell renormalization scheme, we adjust the Z's so that

$$m = M_{\text{physical}} \ \text{ i.e. } \ \Sigma(m^2) = 0$$

and the residue of the pole is one

$$\Sigma'(m^2) = 0 \ .$$

C. Renormalization of Coupling in $\lambda\phi^4$ Theory

Recall that at the one-loop level the 4-point Green function is

$$\begin{aligned} G_4^{(1)} &= i(M_1 + M_2 + M3) \\ &= i\frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[3\left(\frac{1}{\epsilon} - \gamma + \log(4\pi)\right) + F_4\right] + \mathcal{O}(\epsilon) \\ &= i\frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[3\Delta_\epsilon + F_4\right] + \mathcal{O}(\epsilon)\,. \end{aligned}$$

To the order of the λ^2, the 4-point Green function becomes

$$
\begin{aligned}
G_4 &= G_4^{(0)} + G_4^{(1)} \; = \; i\mathcal{M} \\
&= -i\lambda + i\frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[3\Delta_\epsilon + F_4\right] + \mathcal{O}(\epsilon) \quad \text{with} \\
\Delta_\epsilon &= \frac{1}{\epsilon} - \gamma + \log(4\pi)\,, \quad \text{and} \\
F_4 &= +3\log\left(\frac{\mu^2}{m^2}\right) - \int_0^1 dx\,\log\left(1 + \frac{-sx(1-x)}{m^2}\right) \\
&\quad - \int_0^1 dx\,\log\left(1 + \frac{-tx(1-x)}{m^2}\right) - \int_0^1 dx\,\log\left(1 + \frac{-ux(1-x)}{m^2}\right).
\end{aligned}
$$

Integrating over x, we obtain

$$F_4 = 6 + 3\log\left(\frac{\mu^2}{m^2}\right) - \sqrt{1-\frac{4m^2}{s}}\log\frac{\sqrt{1-\frac{4m^2}{s}}+1}{\sqrt{1-\frac{4m^2}{s}}-1}$$

$$-\sqrt{1-\frac{4m^2}{t}}\log\frac{\sqrt{1-\frac{4m^2}{t}}+1}{\sqrt{1-\frac{4m^2}{t}}-1} - \sqrt{1-\frac{4m^2}{u}}\log\frac{\sqrt{1-\frac{4m^2}{u}}+1}{\sqrt{1-\frac{4m^2}{u}}-1}\,.$$

Then the renormalized vertex function becomes

$$iM = -i\lambda + (-i\lambda)^2[iV(s) + iV(t) + iV(u)] - i\delta\lambda\,.$$

Applying the renormalization condition for the vertex function, we need this amplitude to be equal to $-i\lambda$ at zero momentum with $s = 4m^2$ and $t = u = 0$. Therefore, we must set

$$\delta\lambda = -\lambda^2[V(4m^2 + 2V(0)]$$

That determines the counter term of the coupling constant in the on-shell scheme as

$$\delta\lambda = \frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[3\Delta_\epsilon + 3\log\left(\frac{\mu^2}{m^2}\right) - \log\left(1-4x(1-x)\right)\right].$$

Then the finite result becomes

$$\begin{aligned} i\mathcal{M} &= -i\lambda - i\frac{\lambda^2}{32\pi^2}\int_0^1 dx\, F(s,t,u) \\ F(s,t,u) &= \log\left(\frac{-x(1-x)s+m^2}{-x(1-x)(4m^2)+m^2}\right) + \log\left(\frac{-x(1-x)t+m^2}{m^2}\right) + \log\left(\right. \end{aligned}$$

with on-shell renormalization.