PHYS 6213: Advanced Particle Physics, Spring 2022

Lecture 23, Apr 11, 2022 (Monday)

- Reading:
  (a) Chap 12 in Collider Physics
  (b) Chap 18 and Chap 19 in Quantum Field Theory
- Assignments:
  (a) Problem Set 4 due Apr 12 (Tue)

Topics for Today:

Chapter 12 Dimensional Regularization and Renormalization

12.1 Loop Integral in N Dimensions

12.2 Dimensional Regularization

12.3 Mass Renormalization in $\lambda\phi^4$

Topics for Next Lecture:

12.4 Coupling Renormalization in $\lambda\phi^4$

12.1 Loop Integral in N Dimensions

Let's consider an integral in N dimensions[a]

$$\begin{aligned} I_N &= \int d^N\ell F(\ell^2) \\ d^N\ell &= |\ell|^{N-1} d|\ell| d\phi \sin\theta_1 d\theta_1 \sin^2\theta_2 d\theta_2 ... \sin^{N-2}\theta_{N-2} d\theta_{N-2}, \end{aligned} \quad (1)$$

with the following values for the integration variables

$$\begin{aligned} 0 &\leq |\ell| \leq \infty, \\ 0 &\leq \phi \leq 2\pi, \\ 0 &\leq \theta_i \leq \pi,\ i = 1, ..., N-2. \end{aligned} \quad (2)$$

Applying the well known formula

$$\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} = \frac{1}{2} \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \frac{1}{2} B(m,n)\ , \quad (3)$$

[a] *Field Theory: A Modern Primer* by Pierre Ramond

for Re α, Re $\beta > 0$, and $\Gamma(1/2) = \sqrt{\pi}$, we obtain

$$I_N = \frac{\pi^{N/2}}{\Gamma(N/2)} \int_0^\infty dx x^{(N-2)/2} F(x), \tag{4}$$

where $x = |\ell|^2 = \ell^2$.

In general, $F(x)$ will be of the form

$$F(x) = (x + M^2)^{-A} \ , \quad A = 1, 2, \cdots$$

that leads to

$$\begin{aligned} & \int_0^\infty dx\, x^{(N-2)/2} (x^2 + M^2)^{-A} \\ = \ & (M^2)^{-A+N/2} \int_0^\infty dy\, y^{(N-2)/2} (1+y)^{-A} \\ = \ & (M^2)^{-A+N/2} B(N/2, A - N/2) \ = \ (M^2)^{-A+N/2} \frac{\Gamma(N/2)\Gamma(A-N/2)}{\Gamma(A)} \end{aligned}$$

which is valid for Re$(N/2) > 0$ and Re$(A - N/2) > 0$.

Then, we have

$$\int \frac{d^N \ell}{(\ell^2 + M^2)^A} = \pi^{N/2} \frac{\Gamma(A - N/2)}{\Gamma(A)} \times \frac{1}{(M^2)^{A-N/2}} \ . \tag{5}$$

We have derived this expression for N integer, Re $(A - N/2) > 0$, and Re $(N/2) > 0$. Now we can generalize it for non-integer N by analytic continuation.

Now, by letting $q = \ell + p$ we can write Eq. (5) in the following form,

$$\int \frac{d^N \ell}{(\ell^2 + 2\ell \cdot p + M^2)^A} = \pi^{N/2} \frac{\Gamma(A - N/2)}{\Gamma(A)} \frac{1}{(M^2 - p^2)^{A-N/2}} \ .$$

Next by successive differentiation of previous equation with respect to p_μ, it is easy to obtain the formula

$$\int d^N\ell \frac{\ell_\mu}{(\ell^2 + 2\ell \cdot p + M^2)^A} = \pi^{N/2} \frac{\Gamma(A - N/2)}{\Gamma(A)} \frac{(-2p_\mu)}{(M^2 - p^2)^{A-N/2}}$$

and

$$\begin{aligned} & \int d^N\ell \, \frac{\ell_\mu \ell_\nu}{(\ell^2 + 2\ell \cdot p + M^2)^A} \\ = \quad & \frac{\pi^{N/2}}{\Gamma(A)(M^2 - p^2)^{A-N/2}} \\ & \times \left[\Gamma(A - N/2) p_\mu p_\nu + \frac{1}{2} \delta_{\mu\nu} \Gamma(A - 1 - N/2)(M^2 - p^2) \right] \, . \end{aligned}$$

For N = 4, the integral I_N becomes

$$I = \pi^2 \int_0^\infty d\ell^2 \ell^2 F(\ell^2) \, . \qquad (6)$$

Representations of Beta Function

Here are some useful formulas for the beta function:

$$
\begin{aligned}
B(m,n) &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \\
&= 2\int_0^{\pi/2} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\, d\theta \\
&= \int_0^1 x^{m-1}(1-x)^{n-1}\, dx \\
&= \int_0^\infty u^{m-1}(1+u)^{-(m+n)}\, du\ .
\end{aligned}
$$

The Gamma Functions

The following formulae are very useful for dimensional regularization.

$$
\begin{aligned}
\Gamma(-n+\epsilon) &= \frac{(-1)^n}{n!}[\frac{1}{\epsilon}+\psi_1(n+1)+O(\epsilon)] \\
\Gamma(\epsilon) &= \frac{1}{\epsilon}-\gamma+O(\epsilon) \\
\Gamma(-1+\epsilon) &= -[\frac{1}{\epsilon}+1-\gamma+O(\epsilon)] \\
\Gamma(-2+\epsilon) &= \frac{1}{2}[\frac{1}{\epsilon}+1+\frac{1}{2}-\gamma+O(\epsilon)] \qquad (7)
\end{aligned}
$$

where γ is the Euler constant, and

$$
\begin{aligned}
\psi_1(z) &= \frac{d[\ln\Gamma(z)]}{dz} = \frac{\Gamma'(z)}{\Gamma(z)} \\
\psi_1(n+1) &= 1+\frac{1}{2}+\ldots+\frac{1}{n}-\gamma \qquad (8)
\end{aligned}
$$

Feynman Parametrization for Loop Integrals

$$\frac{1}{d_1 d_2} = \Gamma(2) \int_0^1 dx \frac{1}{[x(d_2 - d_1) + d_1]^2}$$

$$\frac{1}{d_1 d_2 d_3} = \Gamma(3) \int_0^1 dx \int_0^x dy \frac{1}{[x(d_2 - d_1) + y(d_3 - d_2) + d_1]^3}$$

$$= \Gamma(3) \int_0^1 dx \int_0^x dy \frac{1}{[x(d_2 - d_3) + y(d_1 - d_2) + d_3]^3}$$

$$\frac{1}{d_1 d_2 d_3 d_4}$$

$$= \int_0^1 dx \int_0^x dy \int_0^y dz \frac{\Gamma(4)}{[x(d_2 - d_1) + y(d_3 - d_2) + z(d_4 - d_3) + d_1]^4}$$

$$= \int_0^1 dx \int_0^x dy \int_0^y dz \frac{\Gamma(4)}{[x(d_3 - d_4) + y(d_2 - d_3) + z(d_1 - d_2) + d_4]^4} .$$

12.2 Dimensional Regularization

In high energy theory, there are 3 types of divergences: (a) ultraviolet ($E \to \infty$), (b) infrared (($E \to \epsilon \to 0+$), and (c) collinear divergence ($\cos\theta \to \pm 1$) between a quark and a gluon:

(a) ultraviolet divergence can be removed by renormalization with a hign energy cut off or dimensional regularization;

(b) infrared divergence can be removed with real gluon or photon emission and a low energy cut off or dimensional regularization;

(c) collinear divergence can be removed by redefinition of parton distribution functions.

Regularization is the procedure to isolate divergences and determine the finite part for physical observables.

Example 1: For mass renormalization, we have

$$\Gamma(-1+\epsilon) = -[\frac{1}{\epsilon} + 1 - \gamma + O(\epsilon)]\,.$$

Example 2: For coupling renormalization, we have

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon)\ .$$

12.3 Mass Renormalization in $\lambda\phi^4$ Theory

In N dimensions with $N = 4 - 2\epsilon$, the Lagrangian density for the ϕ^4 theory becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\mu^{2\epsilon}\phi^4 \,.$$

Applying Feynman rules, we obtain the self energy as a one-loop integral

$$\begin{aligned} -i\Sigma(m^2) &= S(-i\lambda)\mu^{2\epsilon}\int \frac{d^N\ell}{(2\pi)^N}\frac{i}{\ell^2 - m^2 + i\epsilon} \\ &= \frac{1}{2}(-i\lambda)\mu^{2\epsilon}\int \frac{d^N\ell}{(2\pi)^N}\frac{i}{\ell^2 - m^2 + i\epsilon} \end{aligned}$$

where S is the symmetry factor

$$S = \frac{4\cdot 3}{4!} = \frac{1}{2}\,.$$

That leads to

$$\Sigma(m^2) = \frac{1}{2}(i\lambda)\mu^{2\epsilon} \int \frac{d^N\ell}{(2\pi)^N} \frac{1}{\ell^2 - m^2 + i\epsilon} \, .$$

Introducing Wick rotation with $\ell_M^0 = i\ell_E^N$ and $\ell_M^2 = -\ell_E^2 = -\ell^2$, we obtain

$$\begin{aligned}\Sigma(m^2) &= \frac{1}{2}(i\lambda)\mu^{2\epsilon}(2\pi)^{-N} \int i d^N\ell_E \, \frac{-1}{\ell_E^2 + m^2 - i\epsilon} \\ &= \frac{\lambda}{2}\mu^{2\epsilon}(2\pi)^{-N} \int d^N\ell \, \frac{1}{\ell^2 + m^2 - i\epsilon} \, .\end{aligned}$$

This is similar to the following integral in the N-dimensional Euclidean space,

$$\int d^N\ell \, \frac{1}{(\ell^2 + M^2)^A} = \pi^{N/2} \frac{\Gamma(A - N/2)}{\Gamma(A)} \frac{1}{(M^2)^{A-N/2}}$$

with $A = 1$ and $M^2 = m^2 - i\epsilon$. Thus the self energy becomes

$$\Sigma(m^2) = \frac{\lambda}{2}\mu^{2\epsilon}(2\pi)^{-N}\pi^{N/2}\frac{\Gamma(1-N/2)}{\Gamma(1)}\frac{1}{(m^2-i\epsilon)^{1-N/2}}\,.$$

Choosing $N = 4 - 2\epsilon$, we obtain

$$\begin{aligned}
\Sigma_{\rm loop}(m^2) &= \frac{\lambda}{32\pi^2}m^2\Gamma(-1+\epsilon)\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} \\
&= \frac{\lambda}{32\pi^2}m^2\left[-\frac{1}{\epsilon}-1+\gamma+O(\epsilon)\right]\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} \\
&= \frac{\lambda}{32\pi^2}(m^2)\left[-\Delta_\epsilon - 1 + \log\left(\frac{m^2}{\mu^2}\right)+O(\epsilon)\right]\,.
\end{aligned}$$

where

$$\Delta_\epsilon = \frac{1}{\epsilon} - \gamma + \ln(4\pi)\,.$$

We have applied

$$\frac{\pi^{N/2}}{(2\pi)^N} = \frac{\pi^{N/2}}{(4\pi^2)^{N/2}} = (4\pi)^{-(N/2)} = (4\pi)^{-(2-\epsilon)} = \frac{1}{16\pi^2}(4\pi)^{\epsilon}\,,$$

as well as

$$\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} = e^{\ln\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon}} = e^{-\epsilon\ln\left(\frac{m^2}{4\pi\mu^2}\right)} = 1 - \epsilon\ln\left(\frac{m^2}{4\pi\mu^2}\right) + \mathcal{O}(\epsilon^2)\,,$$

and

$$\Gamma(-1+\epsilon) = (-1)\left[\frac{1}{\epsilon} + 1 - \gamma + \mathcal{O}(\epsilon)\right]$$

where γ is the Euler constant and ϵ is an infinitesimal parameter.

12.4 Coupling Renormalization in $\lambda\phi^4$ Theory

In N dimensions with $N = 4 - 2\epsilon$, we need to define a new coupling

$$\lambda_{\rm old} = \lambda_{\rm new} \cdot \mu^{2\epsilon} = \lambda\mu^{2\epsilon}$$

$\lambda = \lambda_{\rm new}$ to keep λ dimensionless so this theory is renormalizable. Then the Lagrangian density becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\mu^{2\epsilon}\phi^4\,.$$

The Next-to-Leading-Order (NLO) vertex corrections have contributions from s, t and $u-$channel diagrams. Applying Feynman rules, we obtain the transition amplitude for the $s-$channel contribution as

$$\begin{aligned} iM_1 &= \frac{1}{2}(-i\lambda)^2(\mu^{2\epsilon})^2 \int \frac{d^N\ell}{(2\pi)^N} \frac{i}{\ell^2 - m^2 + i\epsilon} \frac{i}{(\ell + q_1)^2 - m^2 + i\epsilon} \\ &= \frac{1}{2}\lambda^2(\mu^2)^{2\epsilon}(2\pi)^{-N} \int d^N\ell \, \frac{1}{(\ell^2 - m^2 + i\epsilon)[(\ell + q_1)^2 - m^2 + i\epsilon]} \\ &= \frac{1}{2}\lambda^2(\mu^2)^{2\epsilon}(2\pi)^{-N} \int d^N\ell \, \frac{1}{d_1 d_2} \end{aligned}$$

where

$$\begin{aligned} d_1 &= \ell^2 - m^2 + i\epsilon \\ d_2 &= (\ell + q_1)^2 - m^2 + i\epsilon \end{aligned}$$

with $q_1 = p_1 + p_2$ and $q_1^2 = s$.

Introducing a Feynman parameter x, we obtain

$$\begin{aligned}\frac{1}{d_1 d_2} &= \int_0^1 dx\, \frac{1}{[x(d_2 - d_1) + d_1]^2} \\ &= \int_0^1 dx\, \frac{1}{[\ell^2 + 2\ell\cdot q_1 x - m^2 + x q_1^2)^2}\end{aligned}$$

where

$$d_2 - d_1 = 2\ell \cdot q_1 + q_1^2\,.$$

Then the amplitude becomes

$$\begin{aligned}iM_1 &= \frac{1}{2}(\lambda)^2(\mu^2)^{2\epsilon}(2\pi)^{-N}\int d^N\ell\, \frac{1}{d_1 d_2} \\ &= \frac{1}{2}(\lambda)^2(\mu^2)^{2\epsilon}(2\pi)^{-N}\int d^N\ell \int_0^1 dx\, \frac{1}{[\ell^2 + 2\ell\cdot q_1 x - m^2 + x q_1^2)^2}\,.\end{aligned}$$

In N dimensions with dimensional regularization, we can switch the order of integration and express M_1 as

$$iM_1 = \frac{1}{2}(\lambda)^2(\mu^2)^{2\epsilon}(2\pi)^{-N}\int_0^1 dx\int d^N\ell\,\frac{1}{[\ell^2+2\ell\cdot q_1 x-m^2+xq_1^2)^2}\,.$$

The integral in M_1 can be simplified with a shift $q=\ell+xq_1$ or $\ell=q-xq_1$. Then the amplitude becomes

$$iM_1 = \frac{1}{2}(\lambda)^2(\mu^2)^{2\epsilon}(2\pi)^{-N}\int_0^1 dx\int d^N q\,\frac{1}{[q^2+x(1-x)q_1^2-m^2]^2}\,.$$

Introducing Wick rotation in the complex ℓ^0 plane with $\ell^0 = i\ell_E^N$ and $\ell_E^N\in\mathcal{R}$, we obtain

$$d^N\ell_M = id^N\ell_E \quad \text{and} \quad \ell_M^2 = (\ell^0)^2-|\vec{\ell}|^2 = -(\ell^N)^2-|\vec{\ell}|^2 = -\ell_E^2 = -\ell^2\,.$$

Then the s-channel amplitude becomes

$$iM_1 \;=\; i(-1)^2\frac{1}{2}(\lambda)^2(\mu^2)^{2\epsilon}(2\pi)^{-N}\int_0^1 dx\int d^N q_E\,\frac{1}{[q^2-x(1-x)q_1^2+m^2]^2}$$

or

$$M_1 = \frac{1}{2}(\lambda)^2(\mu^2)^{2\epsilon}(2\pi)^{-N}\int_0^1 dx \int d^N q \, \frac{1}{[q^2 - x(1-x)q_1^2 + m^2]^2} \, .$$

Recall that in the $N-$dimensional Euclidean space, we have

$$I_N = \int d^N \ell \, \frac{1}{(\ell^2 + M^2)^A} = \pi^{N/2}\frac{\Gamma(A - N/2)}{\Gamma(A)}\frac{1}{(M^2)^{A-N/2}} \, .$$

Thus we now have

$$\int d^N q \, \frac{1}{[q^2 - x(1-x)q_1^2 + m^2]^2} \quad = \quad \int d^N q \, \frac{1}{[q^2 + M^2]^2} = \pi^{N/2}\frac{\Gamma(2 - N/2)}{\Gamma(2)}\frac{}{(M}$$

with $A = 2$ and $M^2 = -x(1-x)q_1^2 + m^2$.

Then the s-channel diagram becomes

$$\begin{aligned} M_1 \quad &= \quad \frac{1}{2}(\lambda)^2(\mu^2)^{2\epsilon}(2\pi)^{-N}\int_0^1 dx \, \pi^{N/2}\frac{\Gamma(2-N/2)}{\Gamma(2)}\frac{1}{[-x(1-x)q_1^2 + m^2]^{2-N/2}} \\ &= \quad \frac{1}{2}(\lambda)^2(\mu^2)^{2\epsilon}(2\pi)^{-N}\pi^{N/2}\Gamma(2-N/2)\int_0^1 dx \, \frac{1}{[-x(1-x)q_1^2 + m^2]^{2-N/2}} \end{aligned}$$

with $\Gamma(2) = 1! = 1$.

We often choose $N = 4 - 2\epsilon$. Then $2 - N/2 = \epsilon$, $\Gamma(2 - N/2) = \Gamma(\epsilon)$, and

$$\frac{\pi^{N/2}}{(2\pi)^N} = \frac{\pi^{N/2}}{(4\pi^2)^{N/2}} = (4\pi)^{-(N/2)} = (4\pi)^{-(2-\epsilon)} = \frac{1}{16\pi^2}(4\pi)^{\epsilon}\,.$$

Then the amplitude becomes

$$\begin{aligned} M_1 &= \frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}(\mu^2)^{\epsilon}(4\pi)^{\epsilon})\Gamma(\epsilon)\int_0^1 dx\,[m^2 + x(1-x)q_1^2]^{-\epsilon} \\ &= \frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\Gamma(\epsilon)\int_0^1 dx\,\left[\frac{-x(1-x)q_1^2 + m^2}{4\pi\mu^2}\right]^{-\epsilon}\,. \end{aligned}$$

Now let us consider

$$\Lambda^{-\epsilon} = e^{\ln\left(\Lambda^{-\epsilon}\right)} = e^{-\epsilon\ln(\Lambda)} = 1 - \epsilon\ln(\Lambda) + \mathcal{O}(\epsilon^2)$$

and

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon} + \psi(n+1) + \mathcal{O}(\epsilon)\right]$$

where

$$\begin{aligned}\psi(n+1) &= 1+\frac{1}{2}+\cdots+\frac{1}{n}-\gamma\,, \\ \psi(1) &= -\gamma \simeq -0.5772\,,\end{aligned}$$

γ is the Euler constant and ϵ is an infinitesimal positive parameter $\epsilon \to 0+$. Then the amplitude becomes

$$\begin{aligned}M_1 &= \frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[\frac{1}{\epsilon}-\gamma-\int_0^1 dx\ \ln\left(\frac{m^2-x(1-x)q_1^2}{4\pi\mu^2}\right)+\mathcal{O}(\epsilon)\right] \\ &= \frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[\frac{1}{\epsilon}-\gamma+\ln(4\pi)+\ln\left(\frac{\mu^2}{m^2}\right)-\int_0^1 dx\ \ln\left(1+\frac{-x(1-x)q_1^2}{m^2}\right)\right. \\ &= \frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[\Delta_\epsilon+\ln\left(\frac{\mu^2}{m^2}\right)-\int_0^1 dx\ \ln\left(1+\frac{-q_1^2}{4m^2}4x(1-x)\right)+\mathcal{O}(\epsilon)\right]\end{aligned}$$

where

$$\Delta_\epsilon = \frac{1}{\epsilon}-\gamma+\ln(4\pi)$$

and

$$\ln\left(\frac{m^2 - x(1-x)q_1^2}{4\pi\mu^2}\right) = -\ln(4\pi) - ln\left(\frac{\mu^2}{m^2}\right) + \ln\left(1 + \frac{-q_1^2}{4m^2}4x(1-x)\right) .$$

Applying the following integral

$$\int_0^1 dx \left[1 + \frac{4}{a}x(1-x)\right] = -2 + \sqrt{1+a}\ln\left(\frac{\sqrt{1+a}+1}{\sqrt{1+a}-1}\right) \quad \text{for} \quad a > 0$$

we obtain

$$\int_0^1 dx \ln\left(\frac{m^2 - x(1-x)q_1^2}{4\pi\mu^2}\right) = -\ln(4\pi) - ln\left(\frac{\mu^2}{m^2}\right) + \left[-2 + \sqrt{1 - \frac{4m^2}{q_1^2}}\ln\frac{\sqrt{}}{\sqrt{}}\right.$$

Then the s-channel diagram with $q_1^2 = s$ becomes

$$M_1 == \frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[\frac{1}{\epsilon} - \gamma + \ln(4\pi) + 2 + ln\left(\frac{\mu^2}{m^2}\right) - \sqrt{1 - \frac{4m^2}{s}}\ln\frac{\sqrt{1 - \frac{4m^2}{s}} + 1}{\sqrt{1 - \frac{4m^2}{s}} - 1}\right.$$

Similarly, the $t-$ and $u-$channel diagrams become

$$M_2 = \frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[\frac{1}{\epsilon} - \gamma + \ln(4\pi) + 2 + ln\left(\frac{\mu^2}{m^2}\right) - \sqrt{1-\frac{4m^2}{t}}\,\ln\frac{\sqrt{1-\frac{4m^2}{t}}+}{\sqrt{1-\frac{4m^2}{t}}-}\right.$$

and

$$M_3 = \frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[\frac{1}{\epsilon} - \gamma + \ln(4\pi) + 2 + ln\left(\frac{\mu^2}{m^2}\right) - \sqrt{1-\frac{4m^2}{u}}\,\ln\frac{\sqrt{1-\frac{4m^2}{u}}+}{\sqrt{1-\frac{4m^2}{u}}-}\right.$$

with $q_2^2 = (p_2 + p3)^2 = t$ and $q_3^2 = (p_1 + p3)^2 = u$

Thus the total one-loop contribution at the order of λ^2 is

$$\begin{aligned} G_4^{(1)} &= i(M_1 + M_2 + M3) \\ &= i\frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[3\left(\frac{1}{\epsilon} - \gamma + \ln(4\pi)\right) + F_4\right] + \mathcal{O}(\epsilon) \\ &= i\frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[3\Delta_\epsilon + F_4\right] + \mathcal{O}(\epsilon)\,. \end{aligned}$$

To the order of the λ^2, the 4-point vertex function or the 4-point Green function becomes

$$\begin{aligned} G_4 &= G_4^{(0)} + G_4^{(1)} \\ &= -i\lambda + i\frac{\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[3\Delta_\epsilon + F_4\right] + \mathcal{O}(\epsilon) \quad \text{with} \\ \Delta_\epsilon &= \frac{1}{\epsilon} - \gamma + \ln(4\pi)\,, \quad \text{and} \\ F_4 &= 6 + 3ln\left(\frac{\mu^2}{m^2}\right) - \sqrt{1-\frac{4m^2}{s}}\ln\frac{\sqrt{1-\frac{4m^2}{s}}+1}{\sqrt{1-\frac{4m^2}{s}}-1} \\ &\quad -\sqrt{1-\frac{4m^2}{t}}\ln\frac{\sqrt{1-\frac{4m^2}{t}}+1}{\sqrt{1-\frac{4m^2}{t}}-1} - \sqrt{1-\frac{4m^2}{u}}\ln\frac{\sqrt{1-\frac{4m^2}{u}}+1}{\sqrt{1-\frac{4m^2}{u}}-1}\,. \end{aligned}$$

The divergent part

$$+i\frac{3\lambda^2}{32\pi^2}\mu^{2\epsilon}\left[\frac{1}{\epsilon}\right]$$

needs to be removed by renormalization with counter terms.

Loop Integrals in the Minkowski Space

$$\int \frac{d^N\ell}{i\pi^{N/2}} \frac{1}{(\ell^2 + 2\ell\cdot p + M^2)^A} = \frac{\Gamma(A-N/2)}{\Gamma(A)} \frac{1}{(M^2-p^2)^{A-N/2}} \quad (9)$$

$$\int \frac{d^N\ell}{i\pi^{N/2}} \frac{l_\mu}{(\ell^2 + 2\ell\cdot p + M^2)^A} = \frac{\Gamma(A-N/2)}{\Gamma(A)} \frac{-p_\mu}{(M^2-p^2)^{A-N/2}} \quad (10)$$

$$\int \frac{d^N\ell}{i\pi^{N/2}} \frac{l_\mu l_\nu}{(\ell^2 + 2\ell \cdot p + M^2)^A} = \frac{1}{\Gamma(A)}$$

$$\times [\frac{1}{2}\eta_{\mu\nu} \frac{\Gamma(A-1-N/2)}{(M^2-p^2)^{A-1-N/2}} + p_\mu p_\nu \frac{\Gamma(A-N/2)}{(M^2-p^2)^{A-N/2}}]$$

$$\int \frac{d^N\ell}{i\pi^{N/2}} \frac{l_\mu l_\nu l_\rho}{(\ell^2 + 2\ell \cdot p + M^2)^A} = \frac{-1}{\Gamma(A)}$$

$$\times [\frac{1}{2}(\eta_{\mu\nu}p_\rho + \eta_{\mu\rho}p_\nu + \eta_{\nu\rho}p_\mu) \frac{\Gamma(A-1-N/2)}{(M^2-p^2)^{(A-1-N/2)}} + p_\mu p_\nu p_\rho \frac{\Gamma(A-N/2)}{(M^2-p^2)^{(A-N/}}$$

$$\int \frac{d^N\ell}{i\pi^{N/2}} \frac{l_\mu l_\nu l_\rho l_\sigma}{(\ell^2 + 2\ell \cdot p + M^2)^A} = \frac{1}{\Gamma(A)}$$

$$\times \{\frac{1}{4}(\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma}) \frac{\Gamma(A-2-N/2)}{(M^2-p^2)^{(A-2-N/2)}}$$

$$+\frac{1}{2}(\eta_{\mu\nu}p_\rho p_\sigma + \eta_{\mu\rho}p_\nu p_\sigma + \eta_{\mu\sigma}p_\nu p_\rho + \eta_{\nu\rho}p_\mu p_\sigma + \eta_{\nu\sigma}p_\mu p_\rho + \eta_{\rho\sigma}p_\mu p_\nu) \frac{\Gamma(A -}{(M^2 - p}$$

$$+p_\mu p_\nu p_\rho p_\sigma \frac{\Gamma(A-N/2)}{(M^2-p^2)^{(A-N/2)}}\}$$

All these integrals can be evaluated with dimensional regularization, $N = 4 - 2\epsilon$ and ϵ is infinitesimal.

10.1 12.5 Large Mass Expansion

For $x^2 < 1$, we have

$$(1 \pm x)^{-1} = 1 \mp x + x^2 + O(x^3). \tag{14}$$

For $(q^2/M^2)^2 < 1$, we can expand the propagator[a] as

$$\begin{aligned} [(l+q)^2 + M^2]^{-1} &= [\ell^2 + M^2 + 2\ell \cdot q + q^2]^{-1} \\ &= [\ell^2 + M^2]^{-1}[1 + \frac{2\ell \cdot q + q^2}{\ell^2 + M^2}]^{-1} \\ &= [\ell^2 + M^2]^{-1}[1 - \frac{2\ell \cdot q + q^2}{\ell^2 + M^2} + \frac{(2\ell \cdot q + q^2)^2}{(\ell^2 + M^2)^2} + O(M^{-6})] \end{aligned} \tag{15}$$

[a] M. Veltman, Acta Physica Polonica **B8** (1977) 493.

Three-point Scalar Integral

Now let us calculate the 3-point scalar integral

$$
\begin{aligned}
C_0 &= \int \frac{d^N}{(2\pi)^N} \frac{1}{d_1 d_2 d_3} \\
d_1 &= \ell^2 - m^2 + i\epsilon \\
d_2 &= (\ell + p_1)^2 + i\epsilon \\
d_3 &= (\ell + p_1 + p_2)^2 - m^2 + i\epsilon
\end{aligned}
$$

with $p_1^2 = m^2 = p_2^2$ and $k^2 = (p_1 + p_2)^2 = s$ as an integral in x and y.

Let us introduce Feynman parameters for the denominator of C_0:

$$
\frac{1}{d_1 d_2 d_3} = \Gamma(3) \int_0^1 dx \int_0^x dy \frac{1}{[x(d_2 - d_1) + y(d_3 - d_2) + d_1]^3}
$$

where

$$
\begin{aligned}
d_2 - d_1 &= 2\ell \cdot p_1 + p_1^2 + m^2\,, \\
d_3 - d_2 &= 2(\ell + p_1)\cdot p_2 + p_2^2 - m^2 \;=\; 2\ell\cdot p_2 + 2p_1\cdot p_2 + p_2^2 - m^2\,.
\end{aligned}
$$

Then the denominator becomes

$$
\begin{aligned}
x(d_2 - d_1) + y(d_3 - d_2) + d_1 &= \ell^2 + 2\ell\cdot(xp_1 + yp_2) \\
&\quad + x(p_1^2 + m^2) + y(2p_1\cdot p_2 + p_2^2 - m^2) - m^2 +
\end{aligned}
$$

Applying dimensional regularization, we can switch the order of the ℓ and the x, y integrals and express C_0 as

$$
\begin{aligned}
C_0 &= \frac{\Gamma(3)}{(2\pi)^N}\int_0^1 dx \int_0^x dy \times \\
&\quad \int \frac{d^N\ell}{[\ell^2 + 2\ell\cdot(xp_1 + yp_2) + x(p_1^2 + m^2) + y(2p_1\cdot p_2 + p_2^2 - m^2) - m^2 + i}
\end{aligned}
$$

Let us shift the integration momentum with $q = \ell + xp_1 + yp_2$ or

$\ell = q - xp_1 - yp_2$. Then C_0 becomes

$$C_0 = \frac{\Gamma(3)}{(2\pi)^N} \int_0^1 dx \int_0^x dy \int d^N q \, \frac{1}{[q^2 - M^2]^3}$$

where

$$\begin{aligned} M^2 &= -[-(xp_1 + yp_2)^2 + x(p_1^2 + m^2) + y(2p_1 \cdot p_2 + p_2^2 - m^2) - m^2 + i\epsilon] \\ &= p_1^2 x^2 + p_2^2 y^2 + 2p_1 \cdot p_2 xy - x(p_1^2 + m^2) - y(2p_1 \cdot p_2 + p_2^2 - m^2) + m^2 - \end{aligned}$$

Introducing Wick rotation with $q^0 = iq_E^N$ and $q_M^2 = -q_E^2 = -q^2$, we obtain

$$C_0 = i(-1)^3 \frac{\Gamma(3)}{(2\pi)^N} \int_0^1 dx \int_0^x dy \int d^N q \, \frac{1}{[q^2 + M^2]^3}$$

in the Euclidean space. Recall that in N dimensional Euclidean space

$$\int d^N \ell \, \frac{1}{(\ell^2 + M^2)^A} = \pi^{N/2} \frac{\Gamma(A - N/2)}{\Gamma(A)} \frac{1}{(M^2)^{A-N/2}} \, .$$

For $A = 3$, the 3-point scalar integral becomes

$$\begin{aligned} C_0 &= -i \frac{\Gamma(3)}{(2\pi)^N} \int_0^1 dx \int_0^x dy \, \pi^{N/2} \frac{\Gamma(3 - N/2)}{\Gamma(3)} \frac{1}{(M^2)^{3-N/2}} \\ &= -i \frac{\pi^{N/2}}{(2\pi)^N} \Gamma(3 - N/2) \int_0^1 dx \int_0^x dy \, \frac{1}{(M^2)^{3-N/2}} \end{aligned}$$

where

$$M^2 = p_1^2 x^2 + p_2^2 y^2 + 2p_1 \cdot p_2 xy - x(p_1^2 + m^2) - y(2p_1 \cdot p_2 + p_2^2 - m^2) + m^2 - i\epsilon \, .$$

Choosing $N = 4 - 2\epsilon$, we obtain

$$\Gamma\left(3 - \frac{N}{2}\right) = \Gamma(1 + \epsilon) = 1 + \mathcal{O}(\epsilon) \, .$$

Since it is finite, we may set $N = 4$. Then C_0 becomes

$$C_0 = \frac{-i}{16\pi^2} \int_0^1 dx \int_0^x dy \, \frac{1}{M^2}$$

where

$$M^2 = p_1^2 x^2 + p_2^2 y^2 + 2p_1 \cdot p_2 xy - x(p_1^2 + m^2) - y(2p_1 \cdot p_2 + p_2^2 - m^2) + m^2 - i\epsilon\,.$$

For $p_1^2 = m^2 = p^2$ and $s = (p_1 + p_2)^2 = 2p_1 \cdot p_2 + 2m^2$, we have

$$M^2 = m^2 x^2 + m^2 y^2 + (s - 2m^2)xy - 2m^2 x - (s - 2m^2)y + m^2 - i\epsilon\,.$$

In general, C_0 is expressed in terms of Spence functions.

10.2 The Gamma Functions

The following formulae are very useful for dimensional regularization.

$$\begin{aligned}
\Gamma(-n+\epsilon) &= \frac{(-1)^n}{n!}[\frac{1}{\epsilon}+\psi_1(n+1)+O(\epsilon)] \\
\Gamma(\epsilon) &= \frac{1}{\epsilon}-\gamma+O(\epsilon) \\
\Gamma(-1+\epsilon) &= -[\frac{1}{\epsilon}+1-\gamma+O(\epsilon)] \\
\Gamma(-2+\epsilon) &= \frac{1}{2}[\frac{1}{\epsilon}+1+\frac{1}{2}-\gamma+O(\epsilon)] \qquad (16)
\end{aligned}$$

where γ is the Euler constant, and

$$\begin{aligned}
\psi_1(z) &= \frac{d[\ln\Gamma(z)]}{dz} = \frac{\Gamma'(z)}{\Gamma(z)} \\
\psi_1(n+1) &= 1+\frac{1}{2}+...+\frac{1}{n}-\gamma \qquad (17)
\end{aligned}$$

Divergence Cancellation

Here are some identities for divergence cancellation:

$$\int \frac{d^N\ell}{i\pi^{N/2}} \frac{l_\mu l_\nu}{(\ell^2+M^2)^A} = \frac{1}{\Gamma(A)} \frac{1}{2}\eta_{\mu\nu} \frac{\Gamma(A-1-N/2)}{(M^2)^{A-1-N/2}}$$

$$\int \frac{d^N\ell}{i\pi^{N/2}} \frac{l_\mu l_\nu l_\rho l_\sigma}{(\ell^2+M^2)^A} = \frac{1}{\Gamma(A)} \frac{1}{4}(\eta_{\mu\nu}\eta_{\rho\sigma}+\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\nu\rho}\eta_{\mu\sigma}) \frac{\Gamma(A-2-N/2)}{(M^2)^{(A-2-N/2)}}$$

$$\int \frac{d^N\ell}{i\pi^{N/2}} \frac{1}{(\ell^2+M^2)^2} = \frac{1}{\Gamma(2)} \frac{\Gamma(2-N/2)}{(M^2)^{2-N/2}}$$

$$\int \frac{d^N\ell}{i\pi^{N/2}} \frac{l_\mu l_\nu}{(\ell^2+M^2)^3} = \frac{1}{4}\eta_{\mu\nu} \int \frac{d^N\ell}{i\pi^{N/2}} \frac{(\ell^2+M^2)}{(\ell^2+M^2)^3}$$

$$\int \frac{d^N\ell}{i\pi^{N/2}} \frac{l_\mu l_\nu \ell^2}{(\ell^2+M^2)^4} = \frac{1}{4}\eta_{\mu\nu} \int \frac{d^N\ell}{i\pi^{N/2}} \frac{(\ell^2+M^2)^2}{(\ell^2+M^2)^4} - M^2 \int \frac{d^N\ell}{i\pi^{N/2}} \frac{l_\mu l_\nu}{(\ell^2+M^2)^4}$$

$$\int \frac{d^N\ell}{i\pi^{N/2}} \frac{l_\mu l_\nu l_\rho l_\sigma}{(\ell^2+M^2)^4} = \frac{1}{\Gamma(4)} \frac{1}{4}(\eta_{\mu\nu}\eta_{\rho\sigma}+\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\nu\rho}\eta_{\mu\sigma}) \int \frac{d^N\ell}{i\pi^{N/2}} \frac{(\ell^2+M^2 \ldots}{(\ell^2+M^2 \ldots}$$

10.3 Dirac Matrices

Useful Formulae:

$$
\begin{aligned}
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu &= 2\eta^{\mu\nu} \\
\eta_{\mu\nu}\eta^{\mu\nu} &= N \\
\gamma_\mu \not{a} \gamma^\mu &= -2(1-\epsilon) \not{a} \\
\gamma_\mu \not{a} \not{b} \gamma^\mu &= 4a \cdot b - 2\epsilon \not{a} \not{b} \\
\gamma_\mu \not{a} \not{b} \not{c} \gamma^\mu &= -2 \not{c} \not{b} \not{a} + 2\epsilon \not{a} \not{b} \not{c}
\end{aligned}
$$

where $N = 4 - 2\epsilon$.