PHYS 6213: Advanced Particle Physics, Spring 2022

Lecture 15, Mar 09, 2022 (Wednesday)

- Reading:
  (a) Chap 7 in Collider Physics
  (b) Chap 25–27 in Quantum Field Theory
- Assignments:
  (a) Problem Set 3 due Mar 11 (Fri)
- Make-up Class on Mar 11 (Fri) 09:30 AM–10:30 AM (On Zoom)

Topics for Today:

Chapter 10 Quantum Chromodynamics (QCD)

10.2 QCD Lagrangian and Feynman Rules

10.3 The DeWitt-Faddeev-Popov Formalism

10.4 $e^+e^- \to$ hadrons

Topics for Next Lecture:

10.4 $e^+e^- \to$ hadrons

10.5 The parton model

10.6 The strong coupling parameter

10.2 QCD Lagrangian and Feynman Rules

The QCD Lagrangian can be written as

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\rm YM} + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm DFP} + \mathcal{L}_{\rm M} \\
&= -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi}(\partial_\mu G^{a\mu})^2 \\
&\quad +(\partial_\mu \bar{\chi}^a)\left(\delta^{ab}\partial^\mu + g_s f^{abc} G^{c\mu}\right)\chi^b + \bar{\psi}(i \not{D} + m)\psi \\
&= -\frac{1}{4}(\partial_\mu G^a_\nu - \partial_\nu G^a_\mu)(\partial^\mu G^{a\nu} - \partial^\nu G^{a\mu}) - \frac{1}{2\xi}(\partial_\mu G^{a\mu})^2 \\
&\quad +\partial_\mu \bar{\chi}^a \partial^\mu \chi^a + \bar{\psi}(i \not{\partial} + m)\psi + \mathcal{L}_I \\
\mathcal{L}_I &= +\frac{1}{2} g_s f^{abc}(\partial_\mu G^a_\nu - \partial_\nu G^a_\mu) G^{b\mu} G^{c\nu} - \frac{1}{4} g_s^2 f^{abc} f^{ade} G^b_\mu G^c_\nu G^{d\mu} G^{e\nu} \\
&\quad +g_s f^{abc}(\partial_\mu \bar{\chi}^a) G^{b\mu}\chi^c \\
&\quad -g_s G^a_\mu \left(\bar{\psi}\gamma^\mu T^a \psi\right)
\end{aligned}
$$

where YM = Yang-Mills, GF = gauge fixing, M = matter = quark, and DFP = DeWitt-Faddeev-Popov.

In addition,

- $F^a_{\mu\nu} \equiv \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - g_s f^{abc} G^b_\mu G^c_\nu \ , \quad a = 1, 2, ..., 8,$
- G^a_μ = gluon fields, $a = 1, 2, ..., 8$,
- ψ^i = Dirac spinor fields of quarks, i=1,2,3,
- χ^a = DeWitt-Faddeev-Popov ghost fields, $a = 1, 2, ..., 8$,
- g_s = strong coupling constant,
- T^a = generators of $SU(3)$, $a = 1, 2, ..., 8$, and
- f^{abc} = antisymmetric structure constant of $SU(3)$.

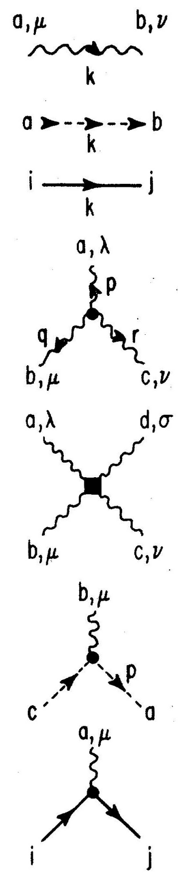
The SU(3) gauge transformation is

$$\psi'(x) = U(\theta)\psi(x) \quad \text{with} \quad U(\theta) = e^{-(i/\hbar)\theta^a(x)T^a}$$

where $T^a = \lambda^a/2$ and λ^a are 3×3 Gell-Mann matrices.

The covariant derivative is $D_\mu = \partial_\mu + i g_s G^a_\mu T^a$.

Feynman Rules



$$-i\delta^{ab}\left[\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)/k^2+\alpha k_\mu k_\nu/k^4\right]$$

$$-i\delta^{ab}/k^2$$

$$i\delta^{ij}\not{k}/k^2$$

$$-gf^{abc}\left[(p-q)_\nu g_{\lambda\mu}+(q-r)_\lambda g_{\mu\nu}+(r-p)_\mu g_{\nu\lambda}\right]$$

$$-ig^2 f^{abe}f^{cde}(g_{\lambda\nu}g_{\mu\sigma}-g_{\lambda\sigma}g_{\mu\nu})$$
$$-ig^2 f^{ace}f^{bde}(g_{\lambda\mu}g_{\nu\sigma}-g_{\lambda\sigma}g_{\mu\nu})$$
$$-ig^2 f^{ade}f^{cbe}(g_{\lambda\nu}g_{\mu\sigma}-g_{\lambda\mu}g_{\sigma\nu})$$

$$gf^{abc}p^\mu$$

$$-ig\gamma^\mu T^a_{ij}$$

Figure 1: QCD Feynman rules.

Gauge boson propagator in the R_ξ gauge

For massive gauge bosons, the propagator is

$$\Delta_{\mu\nu}(k) = \Delta_F(k) = \frac{i\sum_\lambda \epsilon_\mu(k,\lambda)\epsilon_\nu^*(k,\lambda)}{k^2 - M^2 + i\epsilon} = \frac{i[-g_{\mu\nu} + (\xi - 1)k_\mu k_\nu/(k^2 - \xi M)]}{k^2 - M^2 + i\epsilon}\,.$$

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The value of ξ fixes the gauge:

- $\xi = 1$, Feynman gauge,
- $\xi = 0$, Landau gauge,
- $\xi = \infty$, Unitary gauge for massive gauge bosons.

$q\bar{q} \to t\bar{t}$

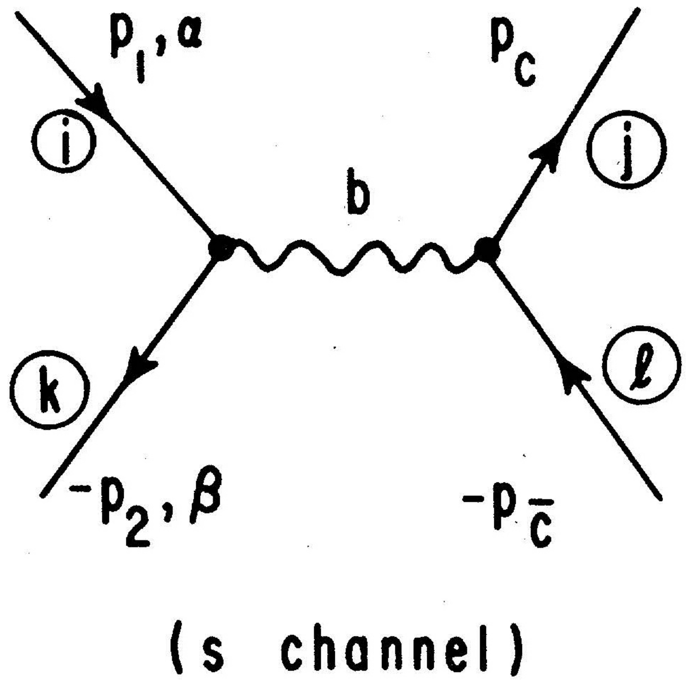


Figure 2: Feynman diagram for $q\bar{q} \to t\bar{t}$.

Gauge Boson Propagator in the Axial Gauge

Gauge invariance leads to unwanted degrees of freedom from self interaction of gauge bosons.

There are two ways that we can remove unwanted degrees of freedom and to make observables physical:

(a) applying the axial gauge, and

(b) introducing spin-0 DeWitt-Faddeev-Popov ghosts that follow Fermi-Dirac statistics,

The axial gauge is defined by the conditions

$$n^\mu G^a_\mu = 0 \quad \text{and} \quad n^2 = n^\mu n_\mu = -1$$

where n is a space-like vector. For example, $n^\mu = (0,0,0,1)$.

The relevant Lagrangian becomes

$$\mathcal{L}_0 + \mathcal{L}_{\rm GF} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi} (n^\mu G^a_\mu)^2 \,.$$

After partial integration, the corresponding quadratic part of the action becomes

$$\frac{1}{2}\int G^{a\mu}\left(\Box g_{\mu\nu} - \partial_\mu\partial_\nu - \frac{1}{\xi}n_\mu n_\nu\right) G^{a\nu}\ d^4x\,.$$

In the momentum space, the operator in brackets becomes

$$\mathcal{D} \to -k^2 g_{\mu\nu} + k_\mu k_\nu - \frac{1}{\xi}n_\mu n_\nu\,.$$

It is straightforward to check that this has the inverse

$$\frac{i}{k^2}\left[-g^{\mu\nu} + \frac{k^\mu n^\nu + n^\mu k^\nu}{n\cdot k} + \frac{(n^2+\xi k^2)k^\mu k^\nu}{(n\cdot k)^2}\right]\,.$$

Note that the two-point Green function for a gluon field is

$$D^{ab}_{\mu\nu}(x-y) \equiv \langle 0|T G^a_\mu(x) G^b_\nu(y)|0\rangle = \int \Delta^{ab}_{\mu\nu}(k)\ e^{-ik\cdot(x-y)}\ \frac{d^4k}{(2\pi)^4}\,.$$

In the limit $\xi \to 0$, we have the axial gauge propagator

$$\Delta^{ab}_{\mu\nu}(k) = \frac{i\delta^{ab}}{k^2 + i\epsilon}\left[-g_{\mu\nu} + \frac{k_\mu n_\nu + n_\mu k_\nu}{n \cdot k} + \frac{n^2 k_\mu k_\nu}{(n \cdot k)^2}\right].$$

It is easy to show that

$$n^\mu \Delta^{ab}_{\mu\nu}(k) = n^\nu \Delta^{ab}_{\mu\nu}(k) = 0$$

and there is no ghost gluon interactions.

In the s-channel diagram for $g(p_1)g(p_2) \to t\bar{t}$ with trilinear gluon interactions, we often choose

$$R^{\mu\nu} = \sum_\lambda \epsilon^\mu(q,\lambda)\epsilon^{*\nu}(q,\lambda) = -g_{\mu\nu} + \frac{2}{s}\left(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu\right)$$

for the polarization sum, where $q = p_1$ or p_2, $k = p_1 + p_2$ and $k^2 = s$.

$gg \to t\bar{t}$

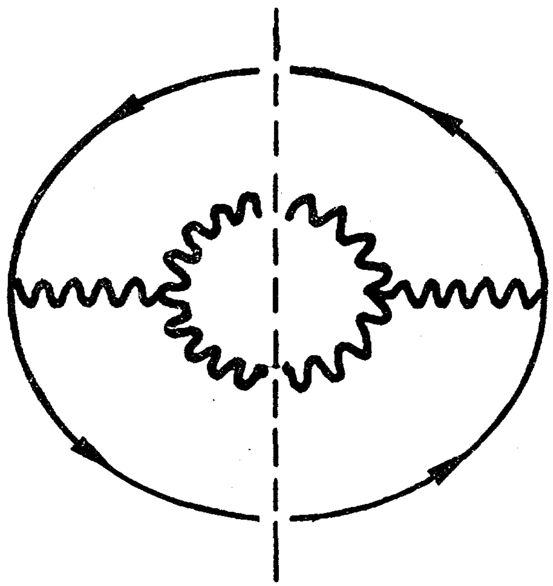


Figure 3: Feynman diagrams for $|M|^2(gg \to t\bar{t})$.

Color Factors

In the fundamental representation, the generators (T^a) are $N \times N$ matrices, and the su(N) algebra is defined by

$$[T^a, T^b] = i f^{abc} T^c \,,$$

where f^{abc} is anti-symmetric structure constants. In addition, we have

$$\{T^a, T^b\} = \frac{\delta^{ab}}{N} + d^{abc} T^c$$

where d^{abc} are symmetric structure constants. Thus

$$T^a T^b = [T^a, T^b] + \{T^a, T^b\} = \frac{1}{2}\left[\frac{\delta^{ab}}{N} + (i f^{abc} + d^{abc}) T^c\right] .$$

With this choice, the color matrices obey the following relations:

$$\mathrm{Tr}(T^a T^b) = T_R \delta^{ab} \,, \quad T_R = \frac{1}{2} \,,$$

and

$$(T^a)_{ml}\,(T^a)_{ln} \quad = \quad C_F\delta_{mn}\,, \quad C_F = \frac{N^2-1}{2N}\,.$$

In matrix formalism, that is

$$T^aT^a = C_F\cdot I \quad \text{and} \quad \mathrm{Tr}(T^aT^a) = C_F\mathrm{Tr}(I)\,.$$

Taking trace on both sides, we have

$$\mathrm{Tr}(T^aT^a) = \frac{1}{2}\delta^{aa} = \frac{N^2-1}{2} = C_F\mathrm{Tr}(I) = C_F\cdot N \quad \text{and} \quad C_F = \frac{N^2-1}{2N}\,.$$

The structure constants have the following sum rules:

$$f^{abc}f^{abd} = C_A\delta^{cd}\,, \quad C_A = N \quad \text{and} \quad d^{abc}d^{abd} = \frac{N^2-4}{N}\delta^{cd}\,.$$

For $SU(3)$, we have

$$C_F = \frac{4}{3} \quad \text{and} \quad C_A = 3\ ,.$$

10.3 The DeWitt-Faddeev-Popov Formalism

A. Yang-Mills Fields

Yang-Mills fields are more complicated than the Maxwell field, because of the non-linear relation

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^1_\mu - g C_{abc} A^b_\mu A^c_\nu \,.$$

This makes the Lagrangian density more complicated

$$\mathcal{L}(x) = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = \mathcal{L}_0(x) + \mathcal{L}_I(x) \,,$$

where the kinetic term is

$$\begin{aligned} \mathcal{L}_0(x) &= -\frac{1}{4} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) \left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}\right) \\ &= -\frac{1}{2} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) \left(\partial^\mu A^{a\nu}\right) \,, \end{aligned}$$

and the interaction Lagrangian becomes

$$\mathcal{L}_I(x) = -gC_{abc}\,(\partial_\mu A^a_\nu)\left(A^b_\mu A^c_\nu\right) - g^2 C_{abc}C_{ade}\left(A^b_\mu A^c_\nu\right)\left(A^{d\mu}A^{e\nu}\right)\,.$$

The kinetic term $\mathcal{L}_0$ contributes to the action in the same form as that of the Maxwell theory

$$\begin{aligned} S_0[A] &\equiv \int d^4x\,\mathcal{L}_0(x) \\ &= \frac{1}{2}\int d^4x\,A_{a\mu}\,(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)\,A_{a\nu}\,. \end{aligned}$$

However, the interaction Lagrangian makes the path integral non-Gaussian and not calculable in a closed form.

One way to proceed is to consider

$$\begin{aligned} Z[J] &= e^{iW[J]} \\ &= N \int \mathcal{D}A e^{iS[A]+i\langle J|A\rangle} G[fA]\Delta[A] \\ &= \exp\left[i \int d^4x\, \mathcal{L}_1 \left(-i\frac{\delta}{\delta J(x)}\right)\right] e^{iW_0[J]} \end{aligned}$$

where

$$e^{iW_0[J]} = N \int \mathcal{D}A e^{iS_0[A]+i\langle J|A\rangle} G[fA]\Delta[A]\,.$$

Let us choose the gauge fixing function as

$$f_a(A) = \partial_\mu A^{a\mu}\,.$$

With the Lorenz condition $\partial_\mu A^{a\mu} = 0$, an infinitesimal gauge transformation on A^a_μ gives

$$\begin{aligned} f_a(A') &= \partial_\mu \left(A^a_\mu + \frac{1}{g}\partial^\mu \eta^a + C_{abc}\epsilon^b \right) A^c_\mu \\ &= \frac{1}{g}\Box\eta^a + C_{abc}(\partial_\mu \epsilon^b) A^c_\mu \, . \end{aligned}$$

Hence

$$\frac{\delta f_a}{\delta \epsilon_b} = \frac{1}{g}\delta_{ab}\Box + C_{abc}\epsilon^b A^c_\mu \partial_\mu \, .$$

and

$$\begin{aligned} \Delta[f(A)] &= \int \mathcal{D}\chi^* \, \mathcal{D}\chi \mathrm{exp} \left[i \int d^4x \, \mathcal{L}_{\mathrm{DFP}}(x) \right] \\ \mathcal{L}_{\mathrm{DFP}}(x) &= \chi^*_a(x) \left[\delta_{ab}\Box + C_{abc}A^c_\mu \partial_\mu \right] \chi^b \, , \end{aligned}$$

where $\chi^*(x)$ and $\chi(x)$ are independent scalar fields obeying Fermi statistics, with anti-commutation and integration rules for Grassmann algebra. They are called DeWitt-Faddeev-Popov ghosts.

Choosing

$$G[f_a] = \exp\left[\frac{i}{2\xi_a}\int d^4x\, f_a^2\right] = \exp\left[\frac{i}{2\xi_a}\int d^4x\, (\partial_\mu A^{a\mu})\right]$$

we obtain

$$Z[J] = \int \mathcal{D}A\,\mathcal{D}\chi^*\,\mathcal{D}\chi \exp\left[i\int d^4x\,(\mathcal{L}_{\text{eff}}(x) + J_{a\mu}A^{a\mu}\right], \quad \text{where}$$

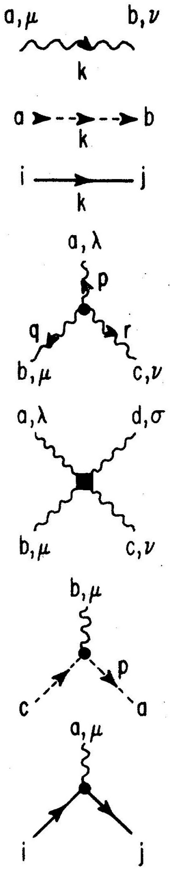
$$\mathcal{L}_{\text{eff}}(x) = \frac{1}{2}A^a_\mu\left[g^{\mu\nu}\Box - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right]A^b_\nu + \frac{1}{2}\chi^{a*}\Box\chi^b + \mathcal{L}_{\text{I}}\,,$$

and

$$\begin{aligned}\mathcal{L}_{\text{I}} \quad = \quad & -gC_{abc}A^{a\mu}A^{b\nu}\partial_\mu A^c_\nu - \frac{1}{4}g^2 C_{abc}C_{ade}A^{b\mu}A^d_\mu A^{c\nu}A^e_\nu \\ & +gC_{abc}\left(\chi^{a*}\partial_\mu\chi^b\right)A^{c\nu}\,.\end{aligned}$$

The system is described as gauge fields with extra interactions to spinless DFP fields with derivative couplings.
Since there is no need to introduce sources for the DFP fields, they only occur in closed loops in Feynman diagrams.

Feynman Rules



$$-i\delta^{ab}\left[\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)/k^2 + a k_\mu k_\nu/k^4\right]$$

$$-i\delta^{ab}/k^2$$

$$i\delta^{ij}\not{k}/k^2$$

$$-gf^{abc}\left[(p-q)_\nu g_{\lambda\mu} + (q-r)_\lambda g_{\mu\nu} + (r-p)_\mu g_{\nu\lambda}\right]$$

$$-ig^2 f^{abe} f^{cde}(g_{\lambda\nu} g_{\mu\sigma} - g_{\lambda\sigma} g_{\mu\nu})$$
$$-ig^2 f^{ace} f^{bde}(g_{\lambda\mu} g_{\nu\sigma} - g_{\lambda\sigma} g_{\mu\nu})$$
$$-ig^2 f^{ade} f^{cbe}(g_{\lambda\nu} g_{\mu\sigma} - g_{\lambda\mu} g_{\sigma\nu})$$

$$gf^{abc}p^\mu$$

$$-ig\gamma^\mu T^a_{ij}$$

Figure 4: QCD Feynman rules.

DeWitt-Faddeev-Popov ghosts in $gg \to t\bar{t}$

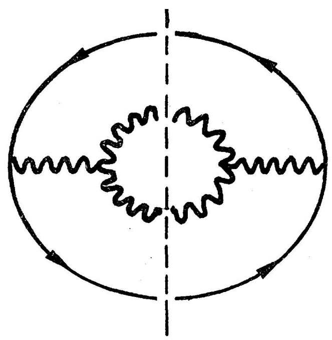


FIG. 25. Gluon loop for polarization sum.

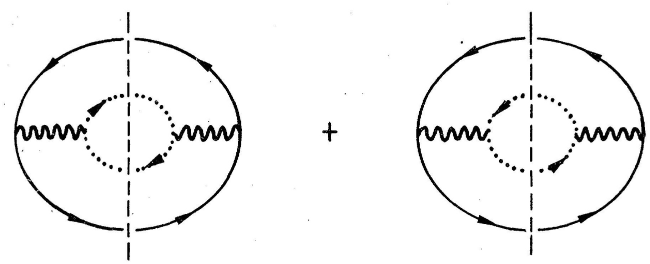


FIG. 26. Ghost loops for polarization sum.

Figure 5: Gluon loop and DFP ghost loops for polarization sums.

B. DeWitt-Faddeev-Popov Formalism in The Weinberg Model

In the Weinberg model, the gauge fixing term is

$$\mathcal{L}_{\rm GF} = -\frac{1}{\xi_W}(f_+ f_-) - \frac{1}{2\xi_Z}(f_Z)^2 - \frac{1}{2\xi_A}(f_A)^2 \,,$$

where

$$\begin{aligned} f_+ &= \partial_\mu W^{+\mu} + i\xi M_W G^+ \,, \\ f_Z &= \partial_\mu Z^\mu + \xi_Z M_Z G^0 \,, \\ f_A &= \partial_\mu A^\mu \,. \end{aligned}$$

The DFP ghost Lagrangian can be derived as

$$\begin{aligned} \mathcal{L}L_{\rm DFP} &= (\partial_\mu \chi_+^*)(\delta W^{+\mu}) + i\xi_W M_W \chi_+^* (\delta G^+) \\ &\quad + (\partial_\mu \chi_-^*)(\delta W^{-\mu}) - i\xi_W M_W \chi_-^* (\delta G^-) \\ &\quad + (\partial_\mu \chi_Z^*)(\delta Z^\mu) + \xi_Z M_Z \chi_Z^* (\delta G^0) + (\partial_\mu \chi_A^*)(\delta A^\mu) \,. \end{aligned}$$

Considering an infinitesimal gauge transformation we can derive the quantities δW^a in terms of gauge parameters ϵ^a

$$
\begin{aligned}
\delta W^{+\mu} &= ig\epsilon^{+}\left(c_W Z^{\mu} + s_W A^{\mu}\right) - ig\left(c_W \epsilon_Z + s_W \epsilon_A\right) W^{+\mu} - \partial_{\mu}\epsilon^{+} \\
\delta Z^{\mu} &= -igc_W \epsilon^{+} W^{-\mu} + igc_W \epsilon^{-} W^{+\mu} - \partial_{\mu}\epsilon_Z \\
\delta A^{\mu} &= -igs_W \epsilon^{+} W^{-\mu} + igs_W \epsilon^{-} W^{+\mu} - \partial_{\mu}\epsilon_A \\
\delta G^{+\mu} &= \frac{i}{2} g\epsilon^{+} G^{0} - \frac{i}{2} g\left(\frac{c_W^2 - s_W^2}{c_W}\epsilon_Z + 2s_W \epsilon_A\right) G^{+} \\
&\quad - \frac{1}{2} gH\epsilon^{+} - \xi M_W \epsilon^{+} \\
\delta G^{0} &= \frac{i}{2} g\left(\epsilon^{+} G^{-} - \epsilon^{-} G^{+}\right) - \frac{g}{2c_W}\epsilon_Z H - \xi_Z M_Z \epsilon_Z \\
\delta H &= \frac{1}{2} g\left(\epsilon^{+} G^{-} + \epsilon^{-} G^{+}\right) + \frac{g}{2c_W}\epsilon_Z G^{0},
\end{aligned}
$$

where $c_W = \cos\theta_W$ and $s_W = \sin\theta_W$.

To derive the DFP Lagrangian, we need to replace ϵ^+ by χ^+, ϵ^- by χ^-, ϵ_Z by χ_Z.

The gauge parameters $\epsilon^+, \epsilon^-, \epsilon_Z$ and ϵ_A are related to the pure $SU(2) \times U(1)$ gauge parameters as follows

$$\begin{aligned} \epsilon^{\pm} &= \sqrt{\frac{1}{2}}\left(\epsilon^1 \mp i\epsilon^2\right) \\ \epsilon_Z &= c_W \epsilon^3 - s_W \epsilon^B\,, \quad \text{and} \\ \epsilon_A &= s_W \epsilon^3 + c_W \epsilon^B\,. \end{aligned}$$

This is similar to the structure of equation for vector bosons.