c. The weighted mean in which the weights are 1, 2, 3, 3, 2, 1, respectively.

18. Ten measurements of the diameter of a hard steel rod with a micrometer caliper yielded the following data:

Diameter, in.

0.250	0.246
0.252	0.250
0.255	0.248
0.249	0.250
0.248	0.252

Calculate the standard deviation and mean deviation of this set of measurements.

19. Show that the error in the *mean* of a series of measurements is always smaller than the largest error in an individual measurement.

20. In a certain set of observations, one observation has a much larger deviation from the mean than the others. If this observation is omitted from the calculations, which measure of spread is affected more, the mean deviation or the standard deviation? Why?

21. If the mean of a large set of observations is m, and all deviations between $-e$ and $+e$ occur equally often, find the mean deviation and standard deviation.

CHAPTER II

PROBABILITY

Any quantitative analysis of random errors of observation must be based on probability theory. It is instructive to consider some simple probability calculations first, as preparation for the task of applying probability theory to the study of random errors.

4 | The Meaning of Probability

If we throw a penny up in the air, we know intuitively that the "chance" of its coming down heads is one-half, or 50%. If we roll an ordinary die (singular of dice) we know that the chance of the number 5 coming up is one-sixth.

What does this really mean, though? On each flip of the penny it comes down either heads or tails; there is no such thing as a penny coming down half heads and half tails. What we really mean is that if we flip the penny a very large number of times, the number of times it comes down heads will be approximately one-half the total number of trials. And, if we roll one die a very large number of times, the number 5 will come up on one-sixth of the trials. For our purposes, it will almost always be most useful to define probabilities in this way. That is, we ask in what fraction of the total number of trials a certain event takes place, if we make a very large number of trials.[1]

It should be pointed out that in the penny-flipping problem we have stated only that the *ratio* of the number of heads to the total number of trials approaches the value one-half as the number of trials becomes very large. This is not the same thing as saying that the *number* of heads approaches the *number* of tails. For example, for 100 flips a fairly probable result is 52 heads. For 10,000 flips a fairly probable result is 5020 heads. In this second case the ratio is much closer to one-half than in the first; yet the *differences* between the number of heads and the number of tails is larger. As a matter of fact, it can be shown that the difference between the number of heads and the number of tails is likely to become very large despite the fact that the ratio of each to the total number of trials approaches one-half. So if you are matching pennies with someone and are losing, you cannot necessarily expect to regain your losses after a sufficiently large number of trials. There is a 50%

[1] A very lucid discussion of some of the basic concepts of probability theory is found in Lindsay and Margenau, "Foundations of Physics," chapter IV, which is available in an inexpensive paperback Dover edition. Many other chapters of this book are also very useful in clarifying some of the basic concepts of physics.

chance that you will lose more and more. But, enough of moralizing.

If we know the probabilities for some simple events, such as tossing a coin or rolling a die, we can calculate probabilities for more complicated events which are composed of these simple events. For example, suppose we flip two pennies at the same time and ask for the probability of getting one head and one tail. When two pennies are flipped, each one can come down in two ways with equal probabilities, so that for two pennies there are four possibilities all together: two heads; heads on the first, tails on the second; tails on the first, heads on the second; or both tails. All four of these possibilities are equally likely, so that we say that each one has a probability ¼. Of the four, two have what we are looking for, namely, one head and one tail. Therefore, the probability of one head and one tail is ½. The probability of two heads is of course ¼, as is the probability of two tails.

Note that these probabilities are always numbers less than 1. If we *add* all the probabilities for *all* the events that can possibly happen, we obtain the total probability that something will happen, which is of course unity.

Here is a slightly more complicated problem. Suppose we roll two dice, the classical number. We ask: What is the probability of rolling 7? Now each die can come down in six positions; so for the two dice there are 36 possible results of rolling two dice, all equally likely. (If we roll n dice, the number of different possibilities is 6^n.) How many of these add up to 7? It is convenient to tabulate the possibilities:

Die 1	*Die* 2
1	6
2	5
3	4
4	3
5	2
6	1

Thus, there are six ways of getting 7 with two dice; the probability for each is $1/36$. The probability of rolling 7 is therefore $6/36$, or $1/6$. In exactly the same way one can show that the probability for 11 is $2/36$, or $1/18$. The probability of rolling either 7 or 11 is the sum of these, $8/36$ or $2/9$, a fact which you may already know.

Note that in the above example, if any of several different events can be regarded as a success, the *total* probability of success is simply the *sum* of the probabilities of the individual events. The situation is a little different if more than one requirement is to be satisfied in order to make the event a success. Suppose, for some strange reason, we rolled two dice and a penny and asked for the probability that the dice will come up totaling 7 and the penny will come up heads. We can look at this problem in two ways. One way is to say that we now have twice as many possibilities as previously because for each of the 36 dice positions there are two positions of the coin, so that we have 72 equally likely

possibilities in all. Still only six of these are favorable; so we say that the probability of success is $6/72$, or $1/12$.

The other, and equally valid, point of view is to recall that the dice rolling and the penny tossing are independent events, each with its own probability. The probability that the dice will come up totaling 7 is $1/6$; the probability that the coin will come up heads is one-half. The probability that *both* these things will happen at the same time is the product of the two probabilities, or $1/12$, in agreement with our other result.

In general, if we are considering several separate and independent events, each with its own probability, the probability that *all* the events will occur is the *product* of the individual probabilities. This fact operates to the advantage of railroads, for example. The probability that a railroad engineer will fall asleep is a small number. The probability that the automatic block-signal system will fail is some other small number. But, for a wreck to take place, both of these things would have to take place at once, and the probability that *both* the engineer will fall asleep *and* the signal system will fail is the product of the two small numbers and is therefore a much smaller number.

To conclude this section, here is another problem of probabilities. Every book concerned with probability contains at least one problem involving drawing black balls and white balls out of an urn; there is no reason why this book should be an exception.

The particular urn we have in mind contains six white balls and four black ones. They cannot be distinguished by touch, and we draw them out without looking. If two balls are drawn out, what is the probability that one is white and the other black, if the first is not replaced before the second is drawn?

Clearly there are two possibilities which we should call successes: white on the first draw and black on the second, and the reverse. Considering the first possibility, we need to multiply the probability of a white ball on the first draw, which is $6/10$, by the probability of a black ball on the second, which is not $4/10$, but $4/9$, since after the first draw the number remaining is 9, of which 4 are black. Thus, the probability of white on the first and black on the second is $(6/10)(4/9) = 24/90 = 4/15$. Similarly, the probability of black on the first and white on the second is $(4/10)(6/9) = 24/90 = 4/15$. The sum of these gives the probability for one white ball and one black one, in either order. This is $4/15 + 4/15 = 8/15$.

The result would have been different if we had replaced the first draw. Then the probability for each case would have been $(6/10)(4/10) = 24/100 = 6/25$, so that the total probability would be $12/25$. Making up more problems as we go along, we note that the probability for two blacks is $(4/10)(3/9)$ if the first ball is not replaced, since after the first draw only three blacks are left if the first ball was black. If the first draw is replaced, then the probability is $(4/10)(4/10)$. And so on.

5 | Permutations and Combinations

The probability calculations in some complicated problems which we will encounter are facilitated considerably

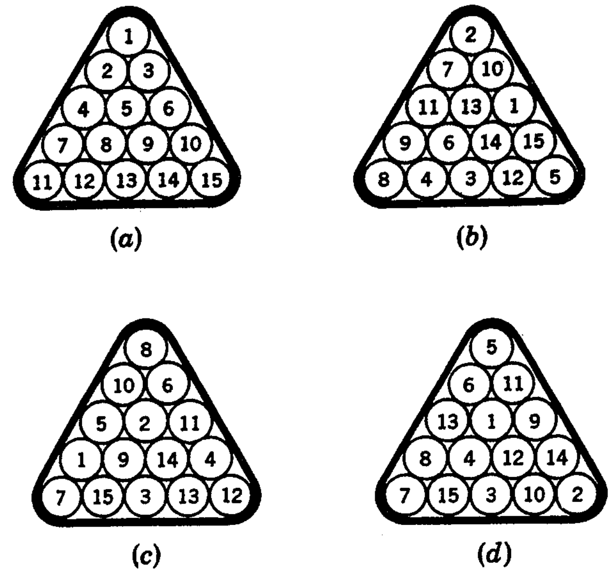


Fig. 5.1. A few of the possible permutations of 15 pool balls. The total number of possible permutations is $15! = 1{,}307{,}674{,}368{,}000$.

by the use of the ideas of permutations and combinations, which we now introduce.

We consider first the idea of permutations of a set of objects. A set of pool balls consists of 15 balls, numbered from 1 to 15. These can be placed in the rack in a number of different ways, not all of which are legal.

In how many different ways can the balls be arranged? Suppose we also number the positions in the rack from 1 to 15, and fill these positions one at a time. To fill the first position we have our choice of any of the 15 balls. For each of these 15 choices there are 14 choices for the second position, because there are 14 balls remaining. For each of these there are 13 choices for the next position, and so forth. Therefore, the number of different ways of filling the entire rack is $(15)(14)(13)(12)\cdots(3)(2)(1)$. Do not bother to multiply out this product; its value is about 1.3×10^{12}, a very large number. The mathematical shorthand for this product is $(15!)$, which is read "15 factorial." In general,

$$N! = N(N-1)(N-2)(N-3)\cdots(4)(3)(2)(1) \tag{5.1}$$

The number of different ways of arranging the 15 objects is called the *number of permutations of* 15 *objects*, and as we have shown this is equal to $15!$. In general, the number of permutations of N objects is $(N!)$.

Next, we consider a slightly different problem, that of selecting a certain number of objects from a group containing a larger number. Let us start with an example. Consider a club containing a total of 10 members. From these members a committee consisting of four members is to be selected. How many different committees can we find?

We can start by choosing one of the 10 members as the first one on the committee; then there are 9 left

to choose for the second member, 8 for the third, and 7 for the fourth. Thus, we might be tempted to say that the total number of possibilities is $(10)(9)(8)(7)$. This, however, is not correct. The reason is that a number of these possibilities would have the same four people, but chosen in various orders. Since we do not care in what order the people are chosen, we do not

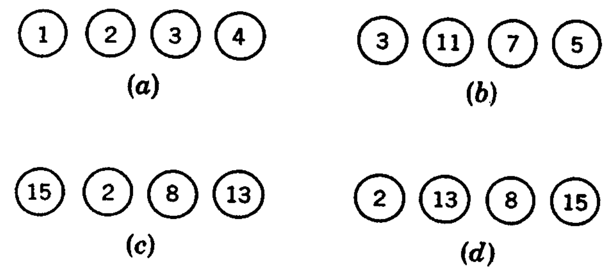


Fig. 5.2. A few of the possible combinations of 15 pool balls taken four at a time. Arrangements (*c*) and (*d*) contain the same balls in different orders; hence they are the *same* combination. The total number of distinct combinations is $15!/11!\,4! = 1365$.

want to count these as different possibilities. Therefore, we must *divide* the above number by the number of ways of rearranging four people, which is simply the number of *permutations* of four objects, $4!$. The correct result for the number of four-man committees which can be chosen from a group of 10 people, if the order of choice is irrelevant, is $(10)(9)(8)(7)/(4)(3)(2)(1)$. This is known as the *number of combinations of* 10 *objects taken four at a time.*

In general, the number of combinations of N things taken n at a time, which we abbreviate $C(N,n)$, is

$$C(N,n) = \frac{N(N-1)(N-2)\cdots(N-n+2)(N-n+1)}{n!} \tag{5.2}$$

This expression can be simplified somewhat by multiplying numerator and denominator by $(N-n)!$. The result is then

$$C(N,n) = \frac{N!}{(N-n)!\,n!} = \binom{N}{n} \tag{5.3}$$

The last expression in Eq. (5.3) is a more common abbreviation for the number of combinations of N things taken n at a time. It is also referred to as a *binomial coefficient*, for reasons which will be explained. From now on, we shall use the notation

$$\binom{N}{n}$$

rather than $C(N,n)$.

In Eq. (5.3) and in other places, we will sometimes encounter cases where $(0!)$ appears, and this has not been defined. How shall we define $(0!)$? The number of combinations of N things taken all N at a time (which is of course just 1) is given by

$$C(N,N) = \binom{N}{N} = \frac{N!}{0!\,N!} = \frac{1}{0!} \tag{5.4}$$

Thus, Eq. (5.3) is correct in all cases only if $0! = 1$. This

is in fact reasonable if we regard $0!$ not as containing a factor zero, but as containing *no factors at all*, so that $0! \times 1 = 1$. We therefore agree on the definition

$$0! = 1 \tag{5.5}$$

The binomial coefficients are, as the name implies, closely related to the binomial theorem. To illustrate the relationship, we consider first a particular example, the expansion of

$$(a+b)^3 = (a+b)(a+b)(a+b)$$

When these three factors are multiplied out, there are terms of the form a^3, a^2b, ab^2, and b^3. The problem is to find the coefficient of each of these terms. First, we note that the only way the term a^3 can appear is from the product in which the factor a is taken in all three of the parentheses. The term a^2b arises from taking a in any two parentheses and b in the third; the number of times this term appears will therefore be the number of different ways that two factors a can be selected from the three parentheses, namely, the number of combinations of three things taken two at a time, which is $3!/2!\,1! = 3$. We therefore find

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

More generally, in expanding a binomial $(a+b)^N$, we note that the expansion is a sum of terms all of which have the form $a^{N-n}b^n$, where n ranges from 0 to N. The number of times this term appears in the expansion is again just the number of combinations of N objects taken n at a time, the "objects" in this case being the b terms in the expansion. We therefore conclude

$$(a+b)^N = \sum_{n=0}^{N} \binom{N}{n} a^{N-n} b^n = \sum_{n=0}^{N} \frac{N!}{(N-n)!\,n!} a^{N-n} b^n \tag{5.6}$$

which is a fancy way of writing the old familiar binomial theorem.

A useful formula for the sum of a set of binomial coefficients can be obtained by placing $a = 1$ and $b = 1$ in this equation. The result is

$$(1+1)^N = 2^N = \sum_{n=0}^{N} \binom{N}{n} \tag{5.7}$$

This may not look particularly useful now, but its usefulness will appear soon.

PROBLEMS

1. A bag contains 10 white marbles and 10 black ones. If three marbles are drawn without looking, what is the probability that all three will be black? Is the situation different if each marble is replaced before the next is drawn?

2. In the game of Russian roulette (not recommended) one inserts one cartridge in a revolver whose capacity is six, spins the chamber, aims at one's head, and pulls the trigger. What is the chance of still being alive after playing the game:

   a. Once?
   b. Twice?

   c. Three times?
   d. A very large number of times?

3. A special pair of dice are marked in the following unorthodox manner: Each die has 1 on three faces, 2 on two faces, and 3 on the remaining face. Find the probabilities for all possible totals when the dice are rolled.

4. Consider a die which, instead of being cubical, is in the shape of a regular tetrahedron (four faces, all equilateral triangles) with numbers 1 to 4. If three such dice are rolled, find the probabilities for all possible totals. Represent the results on a graph.

5. In a group of 30 people selected at random, what is the probability that at least two have the same birthday? Neglect leap years. Solution of this and similar problems is facilitated by use of a log table and an adding machine.

6. Two cards are drawn at random from a 52-card deck. What is the probability that they are the queen of spades and the jack of diamonds?

7. A drawer contains 10 white socks, 10 red ones, and 10 black ones. If their owner arises early and picks out socks in the dark, what is the probability of getting a pair if he picks out two? Three? Four?

8. A carpenter has a tool chest with two compartments, each one having a lock. He has two keys for each lock, and he keeps all four keys on the same ring. His habitual procedure in opening a compartment is to select a key at random and try it. If it fails, he selects one of the remaining three and tries it, and so on. What is the probability that he succeeds on the first try? The second? The third? Would he gain efficiency if he removed one key for each lock, leaving only one of each kind? Explain.

9. In a table of two-digit random numbers, what is the probability that the digit 3 appears exactly once in a two-digit number? Try to make the calculation without listing all the two-digit numbers.

10. A series of observations of the focal length of a lens was made by focusing an image of a distant object (such as the moon) on a screen. The measurements, made to the nearest $\frac{1}{10}$ mm, grouped themselves around the true value with the following probabilities:

Error, mm	*Probability*
0.3	0.04
0.2	0.10
0.1	0.20
0.0	0.25
−0.1	0.20
−0.2	0.10
−0.3	0.04

a. What is the probability that a single measurement will be in error by more than ± 0.15 mm?

b. If three measurements are made, what is the probability that their errors are 0.1, 0.0, and −0.1 mm, respectively?

c. What is the probability that the errors in part *b* will occur, *in any order?*

11. In a batch of 1000 light bulbs, 10% were defective. If a sample of 10 bulbs is taken at random, what is the probability that none of the sample is defective? One? More than one?

12. In Prob. 11, suppose that the percentage of defective bulbs is not known, but in two samples of 10 bulbs each, two were found to be defective in each sample. What conclusions about the total number of defective bulbs can be made?

13. One die is rolled until 1 appears. What is the probability that this will happen on the first roll? The second? The third? The nth? Verify that the sum of these probabilities is unity. (*Hint:* Use the formula for the sum of an infinite geometric progression, $1 + a + a^2 + a^3 + \cdots = 1/(1 - a)$, for $a < 1$.)

14. How many different basketball teams (5 men) can be chosen from a group of 10 men, if each man can play any position?

15. The Explorers' Club has 30 members; an Executive Committee of four is to be chosen. How many possible committees are there?

16. If the Carnegie Tech tennis team has 10 men and the University of Pittsburgh team 7 men, how many different doubles matches between Tech and Pitt can be arranged?

17. How many distinct five-letter words can be formed with the English alphabet, if each word must contain two vowels and three consonants? (There are 5 vowels and 21 consonants in the alphabet.)

18. Four cards are drawn at random from a 52-card deck. What is the probability that they are the four aces?

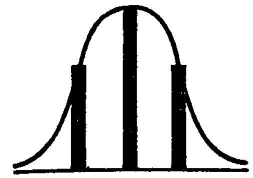
19. In bridge, what is the probability that a certain player will be dealt a hand containing all 13 spades? (Write an expression, but do not carry out the long arithmetic computations.) Is the probability that *someone* at the table will receive this hand the same or different? Explain.

20. In poker, what is the probability of being dealt four of a kind (e.g., four aces, etc.) in a five-card hand? Does this depend on the number of players in the game?

21. Show that if a needle of length a is dropped at random on an array of parallel lines spaced $2a$ apart, the needle lands on a line with probability $1/\pi$.

22. A machine cuts out paper rectangles at random. Each dimension is between 1 and 2 in., but all values between these limits are equally likely. What is the probability that the area of a rectangle is greater than 2 in.^{2}?

CHAPTER III



PROBABILITY DISTRIBUTIONS

We have seen in Sec. 4 how some simple probabilities can be computed from elementary considerations. For more detailed analysis of probability we need to consider more efficient ways of dealing with probabilities of whole classes of events. For this purpose we introduce the concept of a *probability distribution*.

6 | The Meaning of a Probability Distribution

To introduce the idea of a probability distribution, suppose that we flip 10 pennies at the same time. We can compute in an elementary way the probability that four will come down heads and the other six tails. But suppose we ask: What is the probability for the appearance of five heads and five tails, or seven heads and three tails, or more generally, for n heads and $(10 - n)$ tails, where n may be any integer between 0 and 10? The answer to this question is a set of numbers, one for each value of n. These numbers can be thought of as forming a *function* of n, $f(n)$. That is, for each n there is a value