Dice with Parallelepiped Shapes

By Patricia F. Bronson and Robert L. Bronson

Weighting and shaving dice are the usual methods employed to cheat at dice. It is common knowledge that a game played with loaded dice will not have the same outcome as a game played with normal, or unbiased, dice. Dice are loaded by adding small weights just below the surfaces of some of the sides. This weighting changes the probabilities with which certain numbers appear, increasing the probability of rolling the number on the face that is opposite to the face with the weight.[1] The statistics of dice can also be changed by changing their shape from that of a perfect cube. It is for this reason that dice are milled to an accuracy of 0.0025 mm (0.0001 in).

We have constructed dice with the shapes of parallelepipeds not to cheat but to add a new twist to an old game of chance. In this paper we consider not only the change in the probabilities brought about by a change in the shape of dice but present a simple physical model to explain the change. The model is based on the concepts of stable and unstable equilibrium, energy wells, and motions of the center of mass.

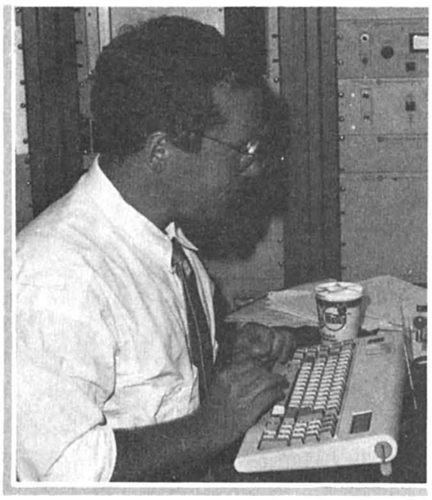
The Problem

When a die is tossed, its initial orientation and velocity are unknown. It hits the ground and tumbles. To write down an equation of motion and solve it seems not only impractical but impossible. Let us approach the problem from another perspective. The die is thrown and it tumbles across the table losing kinetic energy as it goes. At some point it will have lost enough energy so that it can no longer roll over an edge onto the next face but, instead, falls backwards. At this point the die can be considered to be trapped in a specific orientation. We hypothesize that the probability of it coming to rest in a specific orientation is directly proportional to the depth of the potential energy well swept out by its center of mass as it rolls across a face.

With the roll of a normal die, an equal probability of rolling a one through a six is expected. The reason is intuitively obvious, there being no way to distinguish the sides except by their labels. Consider a cube of length ℓ made of a homogeneous material (see top of Fig. 1). The center of mass of the cube lies in the geometrical center, a distance $h = \ell/2$ directly above the center of its base of support. Because of the cubical symmetry, the center of mass will always lie the same distance above the base of support, independent of the side on which the die rests.

Consider now a die that has been deformed into a parallelepiped so that two of its six sides are parallelograms of acute angle θ. The length of a side is still ℓ, but the geometrical center of the die is now closer to the four square faces while remaining the same distance from the faces that are parallelograms. This die is represented by the two drawings in the bottom of Fig. 1. On the lower left side of Fig. 1 we depict the center of mass a distance $h' = h \sin\theta$ from the square surface upon which it rests. Rotating the bottom of this figure toward the observer yields the figure on the lower right side of Fig. 1. From this diagram it is easy to see that the center of mass lies a distance h from the base of support, which is the face that has the shape of a parallelogram.

When resting on a side, the configuration of a die is one of stable equilibrium. That is, when displaced slightly from that position (in this case rotated about an edge) and released, it will return to its original position. If a die is rotated 45° so that the center of mass lies directly above an



Robert L. Bronson *received his B.S. in mathematics and physics from Muhlenberg College in Pennsylvania. He is presently a graduate student in mathematics at the University of California (Santa Barbara, CA 93106).*

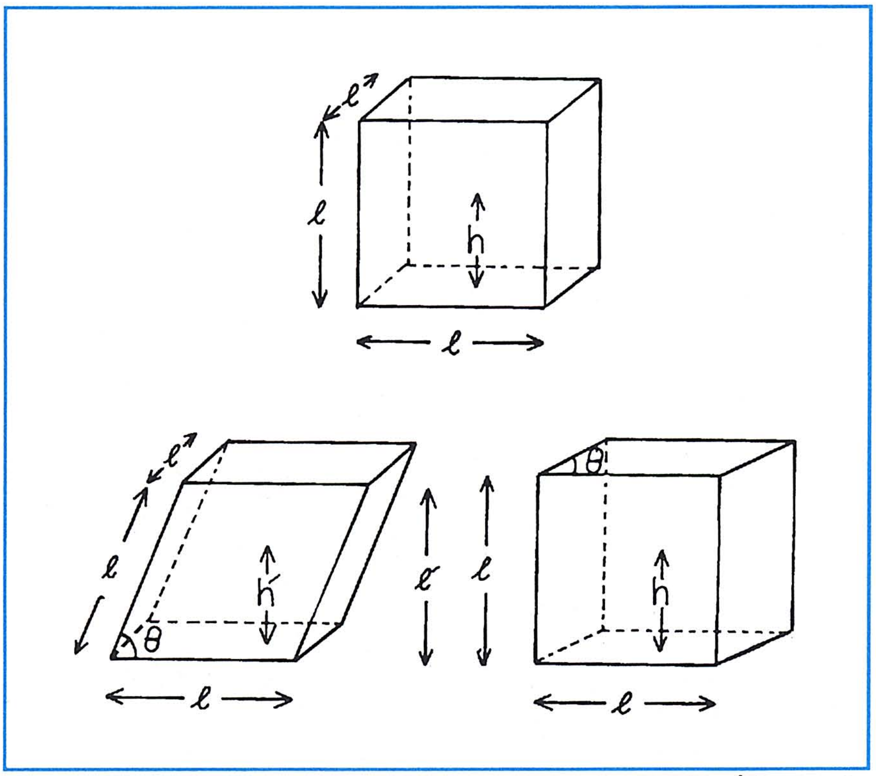


Fig. 1. (Top) A cube of length   has its center of mass a distance $h = \ell/2$ from the center of any face. (Bottom) Two orientations of a parallelepiped, with four faces that are squares and two that are parallelograms, are represented here. The parallelogram is identified by its acute angle θ. In the figure on the lower left the base is ℓ and the height $\ell \sin \theta$. The center of mass lies a distance $h' = h \sin \theta$ from the faces that are square. Rotating the bottom of this figure toward the observer yields the figure on the lower right. From this diagram it is easy to see that the center of mass lies a distance h from the base of support, which is the face that has the shape of a parallelogram.

edge, the configuration is one of unstable equilibrium and with the slightest provocation it could fall either way.[2]

Two cross-sectional diagrams of dice as they roll across a face from edge to opposite edge are represented in Fig. 2. Figure 2 (top) has a cross section that is square, whereas Fig. 2 (bottom) has a cross section in the shape of a parallelogram. In each figure the center of mass is shown to sweep out a curve, which represents the potential energy well for the motion of the die as it rolls across that particular face.[2] It is obvious from these diagrams that the shape of the potential energy wells and their depths (W, W_1, and W_2) depend upon the shape of the cross section. The law of conservation of energy tells us that the die will not get out of the well unless its kinetic energy exceeds its mass times the gravitational constant at the surface of the earth times the depth of the well—mgW. It is our hypothesis that the probability of the die coming to rest on a particular face is proportional to the depth of its potential energy well. We propose verification of this hypothesis by verifying the formula

$$\frac{P'}{P} = k \frac{W'}{W} \qquad (1)$$

where P'/P is the ratio of the probabilities of any two sides, W'/W is the ratio of their well depths, and k a proportionality constant.

The Die and Its Wells

Let us label the faces of the die for reference. Let the sides that are parallelograms be labeled 3 and 4. When the die is resting on either of these sides the center of mass is a distance h above the face in question. Resting on a 3 yields a toss of 4 and vice versa. These sides are distinguishable only by their labels. The four remaining sides (1, 2, 5, and 6) are squares, and the center of mass is a distance h' above the face in question. Rolling a 1 means the die is resting on a 6, rolling a 2 means it's resting on a 5, and vice versa. These four

Patricia F. Bronson *received her Ph.D. in applied physics from Old Dominion University and her B.A. from Adelphi University. She was a professor of physics for five years at Muhlenberg College (Allentown, PA) and is presently a member of the technical staff at the Mission Research Corporation (Santa Barbara, CA 93101).*

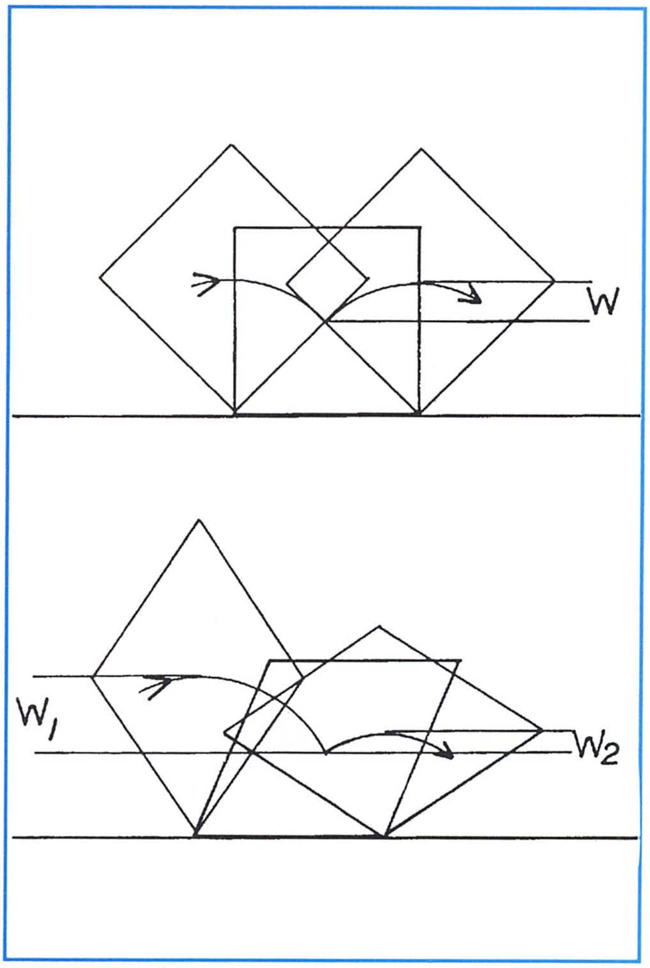


Fig. 2. The potential energy wells swept out by the center of mass as dice roll from edge to face to opposite edge are shown. The wells can be symmetric (top) as in the case of a square, or asymmetric (bottom) as in the case of a parallelogram.

sides are also identical except for their labels. Let the probability of rolling a 3 or a 4 be P (since $P_3 = P_4$) and the probability of rolling a 1, 2, 5, or 6 be P' (since $P_1 = P_2 = P_5 = P_6$).

Three cross sections of the die, taken through its center of mass, are represented in Fig. 3. Note that none of the cross sections are squares, although there are four square faces on the surface of the die. The well of depth W associated with P is a symmetric well across both orientations. The top of the well will be the distance from the edge to the center of mass, labeled r in Fig. 3. Its magnitude is $\sqrt{h^2 + h'^2}$. The well depth W is the difference between r and h

$$W = h\left[\sqrt{1+\sin^2\theta} - 1\right] \quad (2)$$

The well depth W' has one symmetric and one asymmetric component as shown in Fig. 3 (bottom). The components are calculated to be

$$W'_1 = r'_1 - h' = h\sin\theta\left[\frac{1}{\sin(\theta/2)} - 1\right] \quad (3)$$

$$W'_2 = r'_2 - h' = h\sin\theta\left[\frac{1}{\cos(\theta/2)} - 1\right] \quad (4)$$

and

$$W'_3 = r'_3 - h' = h\left[\sqrt{1+\sin^2\theta} - \sin\theta\right] \quad (5)$$

The well depth is the average of these four according to the equation $W' = \left(W'_1 + W'_2 + 2W'_3\right)/4$.

The Experiment and the Ratio of Probabilities

We had three dice constructed using the angles 30°, 45°, and 60° for θ. These dice were milled to an accuracy of 0.0025 mm. According to the Laplace-Gauss theorem, 9604 throws of a suspect die are required to test for a bias to within one percent with 0.95 confidence.[4] Accordingly, each die was rolled 9500 times. The authors, one of whom was a student and thus did most of the tossing, tabulated the results by hand. Our cumulative efforts are presented in Table I. Ideally, we expected $P_1 = P_2 = P_5 = P_6$ and $P_3 = P_4$, but we noticed that the dice showed preference for certain sides and that this preference became more obvious with decreasing angle θ. We attributed this to the fact that as θ decreases so does the volume and mass of the die, making the smaller die more sensitive to minor deviations in measure and homogeneity. Therefore, the ratio P'/P was found by averaging supposedly identical sides. The results, along with those of a perfect cube, are tabulated in Table II. The data P'/P vs W'/W were sent through a least squares fit and are plotted in Fig. 4. The fit is good,

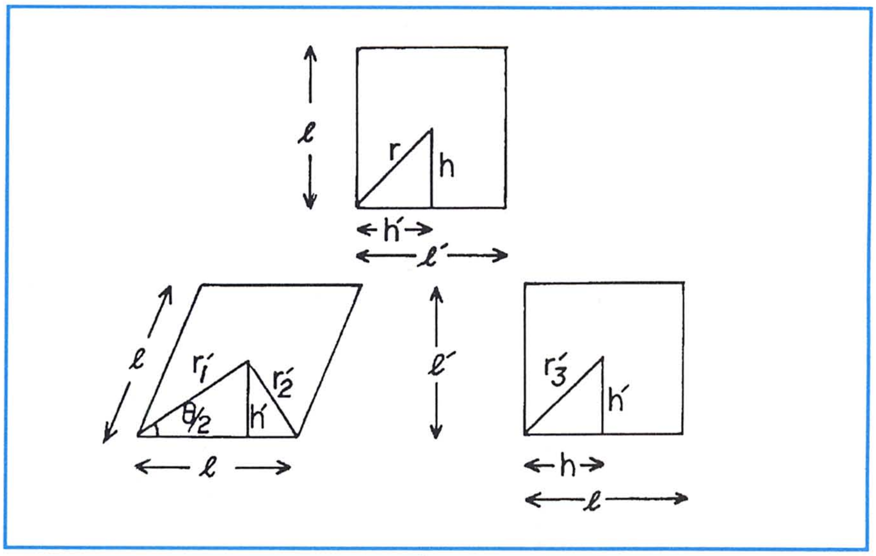


Fig. 3. Geometry for the wells, W and W', in Eqs. (2) through (5). (Top) The well W will have a depth $r - h$. (Bottom) The well W' is found from the average of three different depths $W'_1 = r'_1 - h'$, $W'_2 = r'_2 - h'$, and twice $W'_3 = r'_3 - h'$.

Table I. The results of 9500 throws of the three dice.

θ	1	2	3	4	5	6	total
60°	1890	1914	949	964	1908	1905	9530
45°	2180	2079	493	494	2027	2227	9500
30°	2474	2587	193	215	2123	1908	9500

with an average relative deviation of less than two percent. The linearity of this graph verifies the hypothesis that the ratio of the probabilities is proportional to the well depths, and the slope of the line 2.166 becomes the proportionality constant k in Eq. (1).

Table II. The ratios of the probabilities and well depths.

θ	P'/P	W'/W
90°	1	1
60°	1.991	1.482
45°	4.312	2.485
30°	11.14	5.688

Table III. A table of the probabilities for each die.

θ	K	P	P'
90°	1.00	0.1667	0.1667
60°	1.991	0.1004	0.1998
45°	4.312	0.0520	0.2240
30°	11.14	0.0215	0.2393

The final relationship is $P'/P = 2.166\, W'/W - 1.159$. When $W'/W = 1$ (the case for the normal die) the value of the fitted line's P'/P coordinate is 1.007, exceeding the level of confidence accepted for the given set of statistical data. We attach no significance to points with either coordinate less than 1.

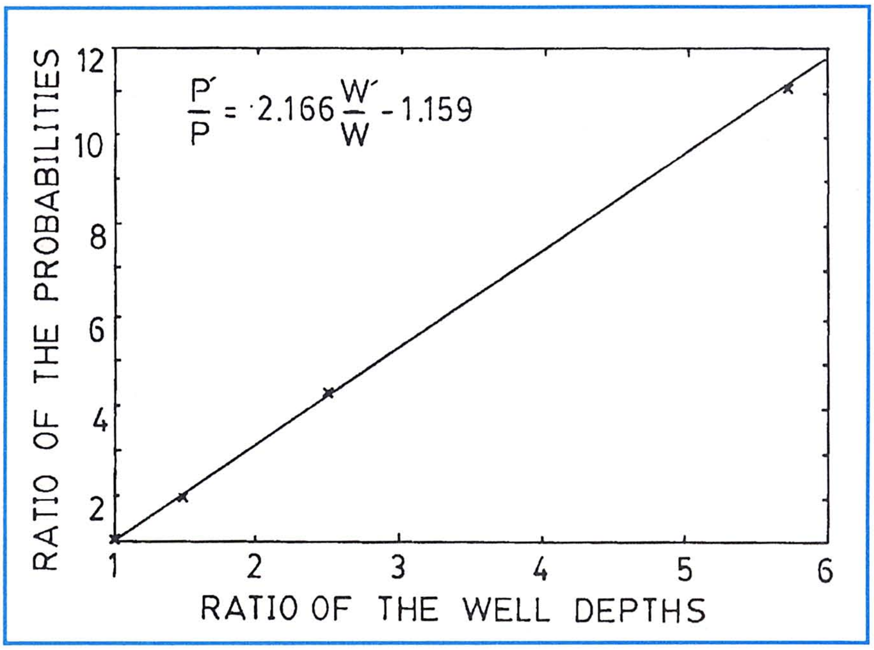


Fig. 4. A graph of the ratio of the probabilities vs the ratio of the well depths is presented. The graph is linear with a slope 2.166 and a y intercept of −1.159.

A Game Played with Matched Biased Dice

Before the probabilities of a game played with two matched irregular dice can be predicted, the probabilities for a single die must be normalized; that is, the sum of all probabilities must be one. For the perfect cube the normalization requires $6P = 1$ or the probability $P = 1/6$. For the parallelepiped, the normalization condition is $4P' + 2P = 1$. Substituting $P' = KP$ into this equation and solving yields P and P'

$$P = \frac{1}{2\,(2K + 1)} \tag{6}$$

and

$$P' = \frac{K}{2\,(2K + 1)} \tag{7}$$

The constant K is just the ratio P'/P in Table II, and the individual probabilities are listed in Table III. For a game played with two die, the probabilities for throwing a 2 through 12 are found by summing the different ways each number can be thrown. In Fig. 5 is a matrix for finding these sums,[4] which are

$$P(2) = P(12) = P'^2 \tag{8}$$

$$P(3) = P(11) = 2P'^2 \tag{9}$$

$$P(4) = P(10) = 2P'P + P'^2 \tag{10}$$

$$P(5) = P(9) = 4P'P \tag{11}$$

$$P(6) = P(8) = 2P'^2 + 2P'P + P^2 \tag{12}$$

and

$$P(7) = 4P'^2 + 2P^2 \tag{13}$$

Figure 6 shows histograms of the probability of an event (throwing a specific number between 2 and 12) vs the event. For the normal dice ($\theta = 90°$), the probability of throwing a 7 is the greatest and the probabilities decrease linearly on both sides. For the $\theta = 60°$ dice, the probability of rolling a 7 is slightly increased, as is the probability of rolling a 2, 3, 11, and 12, while the probability of rolling a 5 or 9 is decreased. An interesting plateau develops on this histogram, showing that the probability of rolling a 3, 4, 5, 9, 10, or 11 are approximately equal. For the $\theta = 45°$

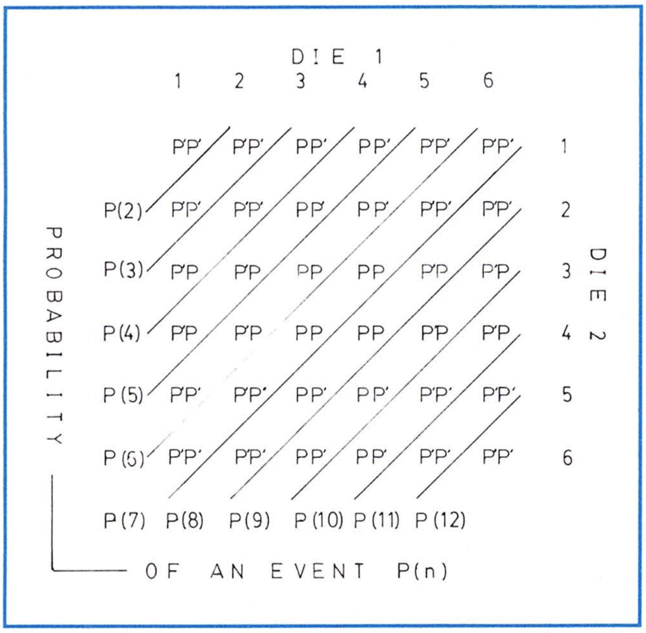


Fig. 5. A matrix for finding the probabilities of rolling a 2 through a 12 using two identically biased die is presented here. The entries in the matrix represent every possible combination of the toss of the dice. The probability of rolling a particular number can be found by summing the terms enclosed in the diagonal lines.

and 30° dice, the relative lowering and raising of the probabilities continues, producing a symmetric distribution about 7 with three distinct peaks. Needless to say, the statistics involved in playing a game with these dice differ greatly from the statistics of playing with traditional cubes.

Conclusions

The hypothesis that the probability of a die being trapped in a specific orientation is directly proportional to the depth of the potential energy well swept out by its center of mass as it rolls across a face was verified by plotting the ratio of the probabilities (found experimentally) to the ratio of the well depths (found theoretically) and finding it to be a straight line.

In conclusion, we believe that this study in the change of the statistics of dice due to a change in their shape provides not only a new twist to an old game but an interesting lesson on stability and potential energy wells. ♦

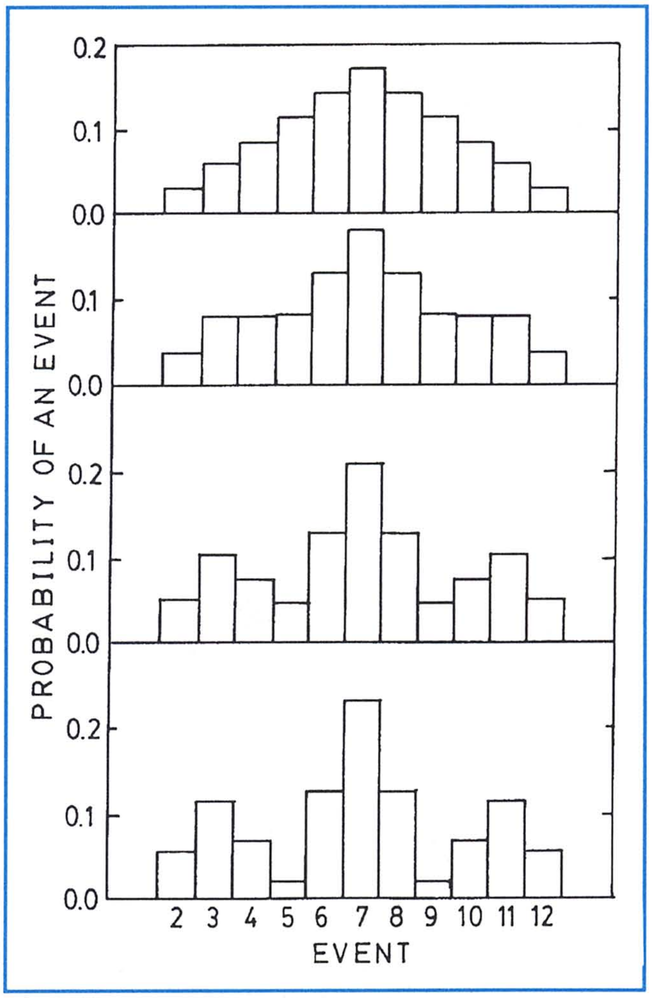


Fig. 6. Histograms of the probability of an event vs the event for two identical die are represented here. Individual histograms are, from top to bottom, the cube, the 60°, 45°, and the 30° dice.

References

1. E.M. Levin, "Experiments with loaded dice," *Am. J. Phys.* **51**, 149 (1983).
2. D. Halliday and R. Resnick, *Fundamentals of Physics*, 3rd ed. (Wiley, New York, 1988), pp. 284–292 and pp. 158–159.
3. Richard A. Epstein, *The Theory of Gambling and Statistical Logic* (Academic Press, New York, San Francisco, and London, 1977).
4. Mary P. Dolciani et al., *Modern Introductory Analysis* (Houghton Mifflin, Boston, 1967), p. 599.