Microcomputers will play an ever increasing role in undergraduate laboratory applications. They provide a means for solving complex differential equations (and other numerical problems), collecting and analyzing data, and demonstrating the overall usefulness of computers in physics. In use for three quarters so far, our microcomputer sequence has been very successful. Via these experiments, introductory physics students show a remarkable grasp of some physical concepts historically considered to be beyond their level. We find we can emphasize the basic "physical" aspects of the pendulum without burying the students in advanced mathematics. Further, in the period–amplitude session, the microcomputer permits a vivid demonstration of the scientific method. The students see a phenomenon theoretically explained, solve the theoretical problem numerically, and then attempt to test the predictive features of the theory through an experiment. In the resistance coefficient session, students see the advantages of using the computer for data collection, storage, and processing. Perhaps most importantly, these experiments serve as an educational example of contemporary technology applied to "old" problems in a new way.

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[1]J. B. Marion, *Classical Dynamics of Particles and Systems*, 2nd ed. (Academic, New York, 1970), pp. 159–165.

[2]These microcomputers are fully described in A. Osborne and C. S. Donahue, *PET/CBM Personal Computer Guide* (Osborne/McGraw-Hill, Berkeley, CA, 1980).

[3]Q is usually defined via the relation $Q^{-1} = \Delta E/(2\pi E)$, where ΔE is the energy loss per cycle and E is the maximum stored energy during the cycle.

[4]The programs used in the experiments were developed at UCSB. In particular, the machine language programs which control input and output to the 6522 VIA timers utilized a few standard sources: A. Osborne and C. S. Donahue, *Versatile Interface Adapter* (Synertek, Santa Clara, CA, 1979) and *MCS 6500 Instruction Set Summary* (MOS Technology, Norristown, PA), and the Commodore PET schematic diagrams.

[5]G. Arfhen, *Mathematical Methods for Physicists* (Academic, New York, 1970), pp. 273–275.

[6]The "midpoint Euler method" is a variation of the standard Euler method commonly used for numerically solving differential equations. [See, for example, R. W. Hornbech, *Numerical Methods* (Quantum, New York, 1975), pp. 189–194.] The variation is affected by calculating the value of the first derivative at the middle of the interval and using this "midpoint value" to calculate the value of the dependent variable at the end of the interval.

[7]Reference 1, pp. 52–55.

[8]b reflects friction due to other sources as well, such as the friction between the pendulum string and the pivot point.

Experiments with loaded dice

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Plastic cubes (dice) were "loaded" for asymmetric insertion of brass cylinders, thereby displacing the mass center from the geometric center of each cube. The statistics of seven differently "loaded" dice have been studied experimentally, and the results analyzed in terms of a Boltzmann-like model involving gravitational energy differences among the various orientations. "Loaded dice" may provide interesting and easily visualized systems for the introduction of concepts such as activation energy, degeneracy, partition function, and elementary group theory.

INTRODUCTION

One hears of gamblers "loading" dice to get an edge from their private knowledge of the altered statistics. In order to explore the effect of mass loading on dice statistics, plastic cubes (dice) (purchased from Commercial Plastics, Inc., 98-31 Jamaica Ave., Richmond Hill, NY 11418) were loaded in various geometries. Results were compared with the predictions of two empirical Boltzmann-like models. The pedagogic value of this investigation is that one encounters calculations involving center of mass, activation energies, and partition functions. The geometric origin of degeneracy is clarified by an examination of the models.

DICE GEOMETRIES

Side and top views of die 1 are shown in Fig. 1, together with parameters of interest. R_1 is the distance between the center of the plastic cube and the center of the cylindrical brass load. The center of mass is displaced from the geometric center by

$$R_{\rm CM} = R_1 V_b(\rho_b - \rho_p)/M, \qquad (1)$$

where V_b, ρ_b are the volume and density of the brass load, respectively, ρ_p is the density of the plastic, and M is the total mass of the loaded die. The gravitational potential energy U of a die is given by $U = MgR$, where R is the height of the center of mass above its lowest value $(L/2 - R_{\rm CM})$.

The gravitational potential energies of die 1, for various orientations, are provided in Fig. 2. The lowest energy configuration [Fig. 2(d)] defines the zero of potential energy, and all energies are calculated with respect to this zero point. After being tossed, a die tumbles until it comes to rest. In this article, a die configuration with energy U_1 will

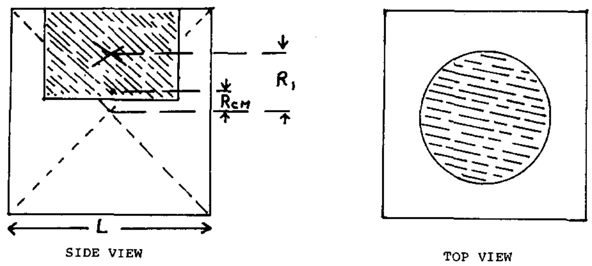

Fig. 1. Side and top views of die 1, with the brass load indicated by shading. R_1 measures the distance from the center of the brass to the geometric center of the cube.

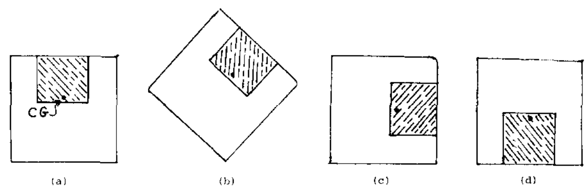

Fig. 2. Die 1 in various states of gravitational potential energy: (a) $U_1 = 128\,000$ erg, (b) $U_1 + \mathrm{EA}_{12} = 207\,900$ erg, (c) $U_2 = 63\,900$ erg, and (d) $U_3 = 0$ erg.

be referred to as state 1. In tumbling from one state to another a gravitational potential barrier must be overcome; there is an activation energy. In Fig. 2(b), die 1 is shown in the highest potential energy state between states U_1 and U_2. The activation energy AE_{12} is given for this case (tumbling from state 1 to state 2) by

$$\mathrm{AE}_{12} = Mg[x - (R_{\mathrm{CM}} + L/2)], \tag{2a}$$

while the activation energy for the reverse tumble is

$$\mathrm{AE}_{21} = Mg(x - L/2), \tag{2b}$$

where

$$x = [(L/2 + R_{\mathrm{CM}})^2 + (L/2)^2]^{1/2}. \tag{2c}$$

Two other parameters, degeneracy and tumbling channels, are needed to complete the geometry of tumbling dice. Degeneracy is the number of states with the same energy. For die 1, the U_1, U_2, U_3 energy levels have degeneracies (g_i) of 1, 4, and 1, respectively. Degeneracies are presented in Table I, for all dice.

The second parameter, the number of tumbling channels (n_{ij}) is the number of ways by which a state with energy U_i can tumble through one 90° rotation, to a state with energy U_j. From a given starting state, there are four possible 90° tumbles into other states. The remaining state is not directly accessible via a 90° rotation. In the terminology of atomic physics, there are four allowed transitions from a given state into adjacent states; the transition to the remaining state is forbidden.[1] For die 1, there are four channels coupling state 1 (energy U_1) to the four states with energy U_2. Thus for die 1 (and 2, 3 also),

$n_{12} = 4$;

$n_{21} = 1$, since only one of the four possible tumbles will flip a state 2 to a state 1;

$n_{23} = 1$, for the same reason as above;

$n_{32} = 4$;

$n_{13} = n_{31} = 0$, since these states do not couple directly through a single 90° rotation;

$n_{22} = 2$, since a state 2 can couple directly with two adjacent degenerate states;

$n_{11} = n_{33} = 0$, since states 1 and 3 are not degenerate.

Dice 2 and 3 are identical to die 1 in all respects except that the brass loadings were smaller. The degeneracies and tumbling channels have the same numerical values as for die 1, since the geometries are identical. Die 4 has an "edge load," with geometric properties quite different from those of the "face-loaded" dice, (numbers 1 to 3). Die 5 has a "corner load"; a brass cylinder lying along a body diagonal. The exposed portion of the brass was ground to a corner configuration. (The determination of the center of mass of this corner load is an interesting problem in itself.) This die has only two energy levels, each threefold degenerate. After tossing die 5, it was modified: three edges were beveled (see Fig. 3) in order to determine whether the reduction in activation energy would affect the statistics measurably. The last "die" tested was a cylinder, with a brass load

Table I. Mechanical properties of dice (cgs units used throughout).

	Die 1	Die 2	Die 3	Die 4	Die 5	Die 6	Die 7
Length of side	3.78	3.78	3.78	3.78	2.53	2.53	2.08 L 2.23 D
Brass length	1.64	1.15	0.85	3.78	1.97	1.97	0.768
Diameter	2.54	2.54	0.637	0.637	0.396	0.396	0.791
Energy U_1(ergs)	128 000	109 000	22 300	82 200	8040	9900	3540
Energy U_2(ergs)	63 900	54 700	11 200	41 100	0	0	1772
n_{11}/EA_{11}	0/	0/	0/	1/94 300	2/14 500	2/5870	0/
n_{12}/EA_{12}	4/79 880	4/68 300	4/52 300	2/66 800	2/9150	2/7860	4/5340
n_{13}/EA_{13}	0/	0/	0/	1/42 610			0/
n_{22}/EA_{22}	2/95 800	2/81 300	2/55 600	0/	2/11 130	2/10 170	2/(20)[a]
n_{23}/EA_{23}	1/63 900	1/54 700	1/47 700	2/50 100			1/4704
$g_1/g_2/g_3$	1/4/1	1/4/1	1/4/1	2/2/2	3/3/0	3/3/0	1/4/1

[a] The densities of plastic and brass were 1.20 and 8.53 g/cm^3, respectively, (20) rather than zero is used in calculations in order to avoid infinites.

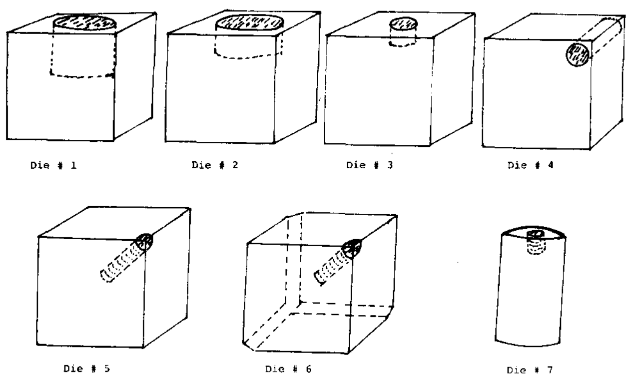

Fig. 3. Sketches of the seven dice.

in one face. This configuration is essentially the limit of die 3, as the activation energy AE_{22} is reduced to zero and the corresponding tumbling reduces to free rolling. The values of all parameters, for each die, are presented in Table I. Their shapes are shown below, in Fig. 3.

EXPERIMENTAL RESULTS

Each die was held by opposite corners and tossed with some top spin, to ensure tumbling and randomization of orientation during flight. A number of surfaces were tested for adequate friction to promote tumbling, and a flat automobile floor mat (corrugated side down) was used. Carpet and other materials into which the edge or corner of a die might sink were deemed unsuitable, as they would affect the activation energies associated with tumbling. Rebounding off a foam rubber backstop at an angle helped to reorient the die, and reduce conservation of its linear and angular momentum. Each die was tossed a minimum of 600 times (generally while watching a football game), and the results tabulated directly with a programmable calculator. These results are presented (Table II) as the fraction of tosses yielding the highest energy state (f_1), and the fraction yielding the next lower energy state (f_2). Where a set of tosses was repeated, all results are shown, as well as the averages of f_1 and f_2.

SEARCH FOR AN EMPIRICAL FORMULA

In seeking an expression that would predict the statistical behavior of these loaded dice, I was guided by the equation for the fraction (f_i) of quantized systems in the ith energy level[2] U_i:

$$f_i = g_i \exp(-U_i/kT)/Z, \tag{3}$$

where Z is the partition function,[3] kT is the thermal energy, and the other symbols are identical to those already introduced. The numerator is the product of the relative probability of finding the system in a state with energy U_i and the degeneracy of that state. The denominator Z is the sum of all such relative probabilities; thus the sum of all f_i equals 1.0. In systems described by Eq. (3), elevated temperature (kT) provides the energy needed for promotion to states with higher energy (U_i).

In our system, initially abundant energy gradually degrades toward zero as the die tumbles. So long as it has a total energy greater than the activation energy for the next tumble, the die is able to continue its motion. When its kinetic and potential energy do not exceed the next activation energy, the die will come to rest. It is clear that activation energy is what enables a die to come to rest in a state of higher potential energy. (A sphere, loaded off center, will always come to rest in its orientation of lowest potential energy.) Specifically, activation energy inhibits the tumbling of a die from a state into an adjacent state.

In modifying Eq. (3), kT cannot simply be replaced by an activation energy. There are four allowed tumbling channels leading out of a given state, and they do not generally have the same activation energies.

Two expressions for f_i that incorporate activation energies are

$$f_i = \frac{g_i}{Z} \exp\left(-\frac{U_i}{4} \sum_j \frac{n_{ij}}{\mathrm{EA}_{ij}} \right) \tag{4}$$

and

$$f_i = \frac{g_i}{Z} \exp\left[-\left(U_i \Big/ {\sum_{j\neq i}}' n_{ij} \right) {\sum_{j \neq i}}' \frac{n_{ij}}{\mathrm{EA}_{ij}} \right]. \tag{5}$$

The difference between the two expressions is that in Eq. (5), tumbling from a state into an isoenergetic state is not included in the sums. Several tests of reasonableness were applied to Eqs. (4) and (5), namely:

(a) If $\mathrm{EA}_{ij} = 0$, where $U_i > U_j$, and $n_{ij} = 0$, then f_i must equal zero. That is, if there is a channel coupling a higher energy state to one of lower energy, with no activation barrier, the die cannot avoid rolling down into the lower energy state. Both equations pass this test.

(b) If all EA_{ij} are infinite, $f_i = g_i/6$. (Six is the limiting

Table II. Values of f_1 and f_2, the fractions of dice throws that yield energy states U_1 and U_2, respectively. Multiple results are shown where experiments were repeated. The predictions of Eqs. (4) and (5) are tabulated. Values of the partition function are also presented.

Experiment	Die 1	Die 2	Die 3	Die 4	Die 5	Die 6	Die 7
f_1/f_2	0.10/0.56	0.09/0.54	0.15/0.62	0.24/0.29	0.33/0.66	0.32/0.68	0.05/0.85
f_1/f_2	0.07/0.57	0.10/0.61	0.085/0.54	0.21/0.26		0.26/0.74	0.07/0.74
f_1/f_2	0.12/0.58	0.08/0.57	0.13/0.64	0.18/0.22		0.24/0.76	0.07/0.79
f_1/f_2	0.055/0.53		0.16/0.62	0.16/0.25		0.25/0.75	0.05/0.80
f_1/f_2	0.07/0.55			0.20/0.26			0.06/0.74
Average f_1/average f_2	0.083/0.56	0.09/0.57	0.13/0.61	0.20/0.26		0.27/0.73	0.06/0.78
Eq. (4) (f_1/f_2)	0.077/0.54	0.076/0.54	0.15/0.63	0.15/0.30	0.33/0.67	0.19/0.81	0.34/0
Eq. (4) (Z)	2.63	2.63	4.51	3.63	4.47	3.69	1.52
Eq. (5) (f_1/f_2)	0.058/0.66	0.06/0.66	0.13/0.67	0.13/0.31	0.29/0.71	0.22/0.78	0.11/0.67
Eq. (5) (Z)	3.50	3.50	4.97	3.56	4.25	3.85	4.55

value of Z.) No tumbling is possible in this case. Both equations pass this test.

(c) If the energy of an upper state (U_i) increases, f_i decreases. This is characteristic of Eqs. (4) and (5), and of course of Eq. (3).

(d) A gedanken experiment appears to eliminate Eq. (4) from consideration. Die 4 has two uppermost states (U_1), which are coupled through a single 90° tumble ($n_{11} = 1$). It is simple to visualize (though not so simple to construct) a die of this geometry, with the edge opposite the loaded edge rounded off so that $EA_{11} = 0$. Equation (4) predicts that $f_1 = 0$ for this case; solely because of a free transition between two upper states. This seems unreasonable, and Eq. (5) was designed to overcome this difficulty.

The predictions of Eqs. (4) and (5) are presented in Table II, below the experimental results. Values of the partition function Z are also presented, as calculated from each equation. Comparing values of f_1 and f_2, calculated from Eqs. (4) and (5), with the average experimental values, it is seen that Eq. (4) is somewhat closer to the experimental values for dice 1 to 5. Equation (5) is a bit closer for die 6, but in the case of die 7, Eq. (4) fails completely, while Eq. (5) shows fair agreement with the experimental results. It may be noteworthy that Eq. (4) provided better agreement with the experimental data for the cases where the dice were cubical, and Eq. (5) where there was deviation from the cubic form.

CONCLUSIONS

Features of this investigation are that it is macroscopic, simple in concept, and easy to carry out. One can readily observe the effect of activation energy, as a die fails to complete a tumble and settles back to rest. The geometric nature of degeneracy, as a consequence of symmetry, is evident. Finally, a simple computer program can be written for a given die, which calculates f_1, f_2, and Z. It is worthwhile and interesting to explore the effect on Z of varying the die geometry.

ACKNOWLEDGMENT

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[1]Many quantum mechanics and atomic physics texts provide an introduction to selection rules. For example, see (a) H. Semat and J. Albright, *Introduction to Atomic and Nuclear Physics*, 5th ed. (Holt, Rinehart & Winston, New York, 1972), p. 299; (b) P. Tipler, *Modern Physics* (Worth, New York, 1978), p. 221.

[2]D. Rapp, *Statistical Mechanics* (Holt, Rinehart & Winston, New York, 1972), p. 17. See also Kittel, *Elementary Statistical Physics* (Wiley, New York, 1958), p. 54, or other texts in *Statistical Mechanics*.

[3]Reference 2, p. 16.

A real "thought" experiment for the hydrogen atom

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The absolute square of the normalized position-space wave function of the electron in a hydrogen atom is interpreted as the probability distribution for observations of the position of the electron. This is only a thought experiment, since the electron's position cannot be observed. The first observation of the momentum distribution of the electron directly verifies the probability interpretation of the momentum-space wave function.

The recent direct measurement by Lohmann and Weigold[1] of the momentum probability distribution of the electron in ground-state atomic hydrogen has significance in the teaching of quantum mechanics. From the very beginning of quantum physics the problem of the hydrogen atom has played a central role. First, it was the subject of daring and far-reaching speculations by Bohr. Later, it was the first problem tackled by Schrödinger[2] with his new wave mechanics and it also emerged as a prime example of the success of quantum mechanics in the first papers on this subject by Heisenberg. Since then it has played a leading role in the teaching of quantum mechanics as well as serving as a most important heuristic tool. Its central importance in the teaching of quantum mechanics is not only due to the fact that the Schrödinger equation can be solved exactly in this case, but also because the solutions form the basis for approximate solutions for other atoms and molecules and the angular solutions are valid for any central field problem. As a heuristic tool it has been used to shape our intuition and to inspire many different expositions.

The solution of the Schrödinger equation for the hydrogen atom is usually carried out using the position representation, and the wave functions $\psi_{nlm}(\mathbf{r})$, which are solutions of the equation, are to be interpreted as probability amplitudes. Born was the first to suggest in 1926 that the square modulus of the wave function represents the probability density of finding the electron at a position r with respect to the center of mass of the atom. This is the simplest physically real quantity that can be derived from ψ. However, although $|\psi_{nlm}(\mathbf{r})|^2$ is stated to be a physical observable, it has never been directly observed. The standard texts all show only calculated values of $|\psi_{nlm}(\mathbf{r})|^2$ and discuss it at most by means of thought experiments. The nearest one can come to a measurement of position information for an atom is to measure the probability distribution for the transfer of momentum to a scattered x ray or electron.[3] This is called the charge form factor. For high enough energy it is essentially the Fourier transform of the electron position distribution summed for all electrons. Except in the case of the hydrogen atom this technique does not allow one to measure information for individual values of n, l, or m.