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DYNAMICS OF AN INVERTED PENDULUM WITH A VIBRATING SUSPENSION

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An inverted (turned around) pendulum is a peculiar phenomenon in mechanics. It is usually unstable at its highest position, since the smallest perturbation disturbs its equilibrium and returns it to its initial stable state. The pendulum becomes stable in this position if the suspension point oscillates (vibrates) with the necessary frequency and amplitude. In this case, the weight undergoes periodic deviations from the vertical position.

The fact that the pendulum stabilizes in the above circumstance is unusual enough to have attracted the attention of a wide range of specialists in physics, mechanics, the theory of vibration, the theory of stability and control, and other fields. At various times, pendulum dynamics have been studied by such noted Soviet and foreign scientists as N. N. Bogolyubov, J. Stoker, P. L. Kapits, K. G. Valeev, and V. N. Cheloma. Detailed investigations of the theory of the vibration of inverted pendulums have been made in recent years by T. G. Strizhak [5, 6]. The present article offers a historical overview of these studies.

In all of the best-known works [5], the dynamics of inverted pendulums has been studied as a closed system. These investigations have reliably established that the stability of the vibrations of such bodies can be characterized by the Mathieu equation, with regions of stability being determined on an Ains–Strett diagram. The qualitative and quantitative aspects of the vibration process have yet to be thoroughly explored. We believe that somewhat more accurate results can be obtained by regarding an inverted pendulum as an open nonequilibrium dynamic system.

The notion of studying individual objects (biological, economic, ecological, social, etc.) and open nonequilibrium systems is associated with the Brussels school of I. Prigozhin [4]. Adherents of this approach have shown that, depending on the conditions, a system either degrades or adapts to a new form of organization (such as oscillation). Prigozhin referred to such open nonequilibrium systems as dissipative structures. These structures are formed and remain intact as a result of an exchange of energy (mass, information) with the environment under nonequilibrium conditions (at branch points [4]).

Mechanical systems in which energy is transferred from sources (by friction, streams of air or gas, vibrations, etc.) to working elements are also open and nonequilibrium. In this article, we study an inverted pendulum with a vibrating suspension point as an open system transmitting vibrational energy (vibrations) to a working element. A study was made previously [1, 2] of open nonequilibrium systems in the form of a Froude pendulum and a rolling mill. In these systems, external energy was transferred from the source to the working element by friction.

1. Figure 1a presents a diagram of the inverted pendulum. It consists of a weightless rigid rod OO_1 having a mass point O_1 of mass m at its end. The suspension point of the pendulum coincides with the origin of the stationary coordinate system XOY. Figure 1b shows a diagram of the vibration of the inverted pendulum. Here, suspension point O moves together with the movable coordinate system X_1OY over the path $\pm l$ at the linear velocity $\pm V$. The character of motion of the suspension point moves in this manner, the mass point O_1, of mass m, will undergo complex motion relative to the movable coordinate system X_1OY.

Proceeding on the basis of the results presented in [5], we can construct a differential equation that describes the complex motion of point O_1. The equation is based on the theorem of the increment of the kinetic energy T_Δ of point O_1 in each cycle of motion of movable (noninertial) coordinate system X_1OY. First we write the increment of the kinetic energy of O_1 as it moves in inertial coordinate system XOY.

$$T_\Delta = G_l \cdot [R(\pi \pm \theta_\Delta) + R \cdot \cos(\pi \pm \theta_\Delta) \pm l], \tag{1.1}$$

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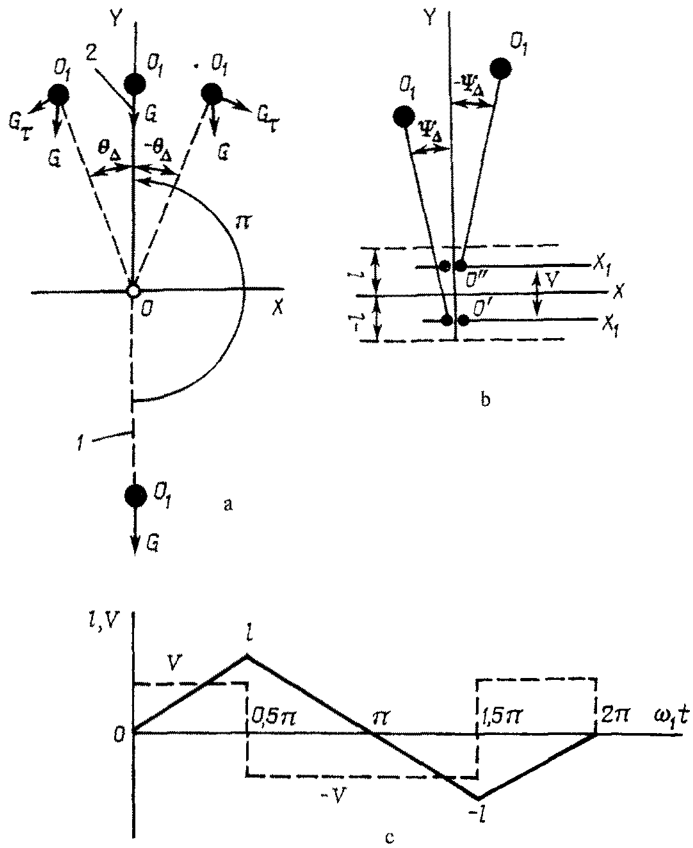

Fig. 1

where $G_\tau = mg \cdot \sin(\pi \pm \theta_\Delta)$ is the tangential component of the force of gravity on point O_1; $R(\pi \pm \theta_\Delta) + R \cdot \cos(\pi \pm \theta_\Delta) \pm l]$, is the displacement vector of O_1.

If we consider that $T_\Delta = (m \cdot R \cdot d^2\theta_\Delta/dt^2) \cdot [R(\pi \pm \theta_\Delta) + R \cdot \cos(\pi \pm \theta_\Delta) \pm l]$, where $F_i = m \cdot R \cdot d^2\theta_\Delta/dt^2$ is the inertial force on point O_1 and the expression in the brackets corresponds to the distance OO_1 (Fig. 1b), and if we then insert the values of T_Δ and G_τ into (1.1), after completing the transformation we obtain

$$\begin{aligned} & mR \cdot [R(\pi + 1) \pm l] \cdot \frac{d^2\theta_\Delta}{dt^2} + \\ & + Rm \frac{d^2\theta_\Delta}{dt^2}(\pm\theta_\Delta) = -mgR(\pi + 1)\sin\theta_\Delta. \end{aligned} \tag{1.2}$$

In Eq. (1.2), we considered that the work of the conservative force G_τ on closed linear path $\pm l$ is zero, as is the work done by this force on the closed arc $+R \cdot \theta_\Delta$. The increment of kinetic energy $R^2 m\, d^2\theta_\Delta/dt^2(\pm\theta_\Delta) = mu \cdot du$ corresponds to the work $A_{T_\Delta} = \eta \cdot mudu$ done by the inertial force F_i on the small closed arc $\pm R\theta_\Delta$, where $u = R \cdot d\theta_\Delta/dt$, where η is the relative viscosity of the medium in which mass point O_1 of mass m is vibrating. Allowing for the fact that $\sin\theta_\Delta \approx \theta_\Delta$, after completing the necessary transformations we obtain

$$\frac{d^2\theta_\Delta}{dt^2} + \frac{\eta V}{R(\pi+1) \pm l} \cdot \frac{d\theta_\Delta}{dt} - \frac{\eta R}{R(\pi+1) \pm l}\left(\frac{d\theta_\Delta}{dt}\right)^2 + \frac{g(\pi+1)}{R(\pi+1) \pm l} \cdot \theta_\Delta = 0 \tag{1.3}$$

or

$$\frac{d^2\theta_\Delta}{dt^2} + 2\beta\frac{d\theta_\Delta}{dt} + \omega_0^2 \cdot \theta_\Delta = \frac{\eta R}{(\pi+1)R \pm l} \cdot \left(\frac{d\theta_\Delta}{dt}\right)^2, \tag{1.4}$$

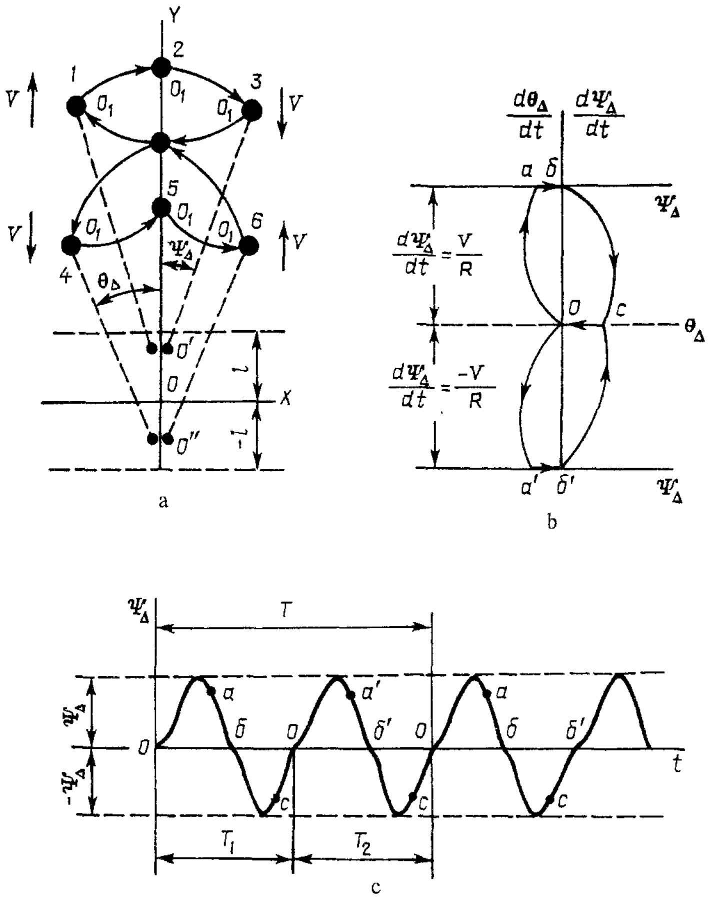

Fig. 2

where

$$\beta = \frac{V \cdot \eta}{R(\pi + 1) \pm l}, \quad \omega_0^2 = \frac{g(\pi + 1)}{R(\pi + 1) \pm l},$$

Here, β and ω_0 are the damping factor and the angular frequency of natural vibration of the bob of the pendulum.

For actual pendulums of the type being discussed, when R $\gg$ $|l|$, the parameters β and ω_0 of differential equation (1.3) are simplified as follows

$$\beta = 0,5 \cdot \frac{V \cdot \eta}{R(\pi + 1)} \qquad \text{and} \qquad \omega_{01}^2 = \frac{g}{R}. \tag{1.5}$$

The bon of the pendulum O_1 undergoes complex periodic motion relative to the moving coordinate system X_1OY during each period of travel of the suspension (Fig. 1b). To describe such motions, it is necessary to determine the vector of linear velocity of the bob $u = R \cdot d\theta_\Delta/dt$, having expressed it in terms of the vector of linear velocity V of the suspension and the vector of relative velocity $u_1 = R \cdot d\psi_\Delta/dt$ of the mass concentrated at point O_1, i.e.,

$$R \cdot \frac{d\theta_\Delta}{dt} = \pm V + R \cdot \frac{d\psi_\Delta}{dt} \qquad \text{for} \qquad \frac{d\theta_\Delta}{dt} = \pm \frac{V}{R} + \frac{d\psi_\Delta}{dt}, \tag{1.6}$$

From this

$$\theta_\Delta = \pm \frac{Vt}{R} + \psi_\Delta. \tag{1.7}$$

Inserting the values of $d\theta_\Delta/dt$ and θ_Δ from (1.6) and (1.7) into (1.3) and considering that

$$u = R\frac{d\theta_\Delta}{dt} = \pm V + R\frac{d\psi_\Delta}{dt},\ du = u_\Delta = \pm V - R\frac{d\theta_\Delta}{dt} = -R\frac{d\psi_\Delta}{dt} \text{ и } \theta_\Delta \approx \sin\theta_\Delta,$$

we obtain Mathieu's differential equation with the right side [5-7]

$$\begin{aligned}\frac{d^2\psi_\Delta}{dt^2} - \frac{V\cdot\eta}{R(\pi+1)\pm l}\cdot\frac{d\psi_\Delta}{dt} + \frac{g(\pi+1)}{R(\pi+1)\pm l}\cdot\cos\omega_1 t\cdot\psi_\Delta = \\ = \frac{R\cdot\eta}{R(\pi+1)\pm l}\left(\frac{d\psi_\Delta}{dt}\right)^2 - \frac{g(\pi+1)}{R(\pi+1)\pm l}\cdot\sin\omega_1 t,\end{aligned} \tag{1.8}$$

where $\omega_1 = V/R$ is the angular frequency of the parametric excitation acting on the mass point O_1 (the bob of the pendulum). Differential equation (1.8) can be changed to the more compact form

$$\frac{d^2\psi_\Delta}{dt^2} - 2\beta\cdot\frac{d\psi_\Delta}{dt} + \omega_0^2\cdot\cos\omega_1 t\cdot\psi_\Delta = \frac{R\cdot\eta}{R(\pi+1)\pm l}\left(\frac{d\psi^2}{dt}\right)^2 - \omega_0^2\cdot\sin\omega\, t. \tag{1.9}$$

2. Differential equations (1.3) and (1.9) characterize the motion of the bob (mass point O_1 of mass m) of the pendulum during each cycle of motion of the coordinate system X_1OY connected with the suspension point (Fig. 1c). The sequence of motion of point O_1 is represented in Fig. 2a in the form of arcs of a graph. The initial arcs 0−1−2 of the graph correspond to the estimate of the phase-plane vector 0−a−b shown in Fig. 2b and obtained from a qualitative analysis of Eq. (1.3) for the case $l > 0$, $V > 0$ (Fig. 1c). In this case, in the inertial coordinate system the bob is accelerated at the velocity $d\theta_\Delta/dt$ by variable positive nonlinear feedback. The pendulum follows the motion of the suspension as it deviates from its previous path. As soon as the angular velocity of the bob (point O_1) reaches $d\theta_\Delta/dt = V/R$, its acceleration is complete and point V/R begins to follow the suspension point at a constant angular velocity V/R. The segment a-b represents this section of movement of the image point of phase-plane vector 0−a−b.

The equilibrium regime in the motion of point O_1 is characterized by the end of segment a−b of the phase-plane vector (Fig. 2b), where $\theta_\Delta \approx 0$ (1.3). This regime lasts only an instant, because the terms $V^2\eta/R[R(\pi + 1) + l]$ in the left and right sides of Eq. (1.3) cancel out. Since the suspension continues to move in accordance with the program (Fig. 1c), an uncontrollable coupling develops and the dynamics of the bob (point O_1) is described by differential equation (1.9).

The subsequent motion of point O_1 is characterized by the direction of the arcs 2−3−0 in Fig. 2a (or the phase-plane vector b−c−0 in Fig. 2b). Here, the body begins to move in a complex manner relative to coordinate system X_1OY. The body initially moves in accordance with b−c−0, and the relative angular velocity $d\psi_\Delta/dt$ is low. Thus, the main perturbing factor in the right side of (1.9) is the term $\omega_0^2\cdot\sin\omega_1 t$. However, an increase in relative angular velocity $d\psi_\Delta/dt$ is accompanied by an increase in the term $(R\cdot\eta/R(\pi + 1) \pm l)(d\psi^2/dt)^2$ in the right side of (1.9), which stabilizes the motion of the body (decreases the angle $\psi_\Delta(t)$). The so-called "standing" regime begins when relative angular velocity $d\psi_\Delta/dt = V/R$ (when $d\theta_\Delta/dt = 0$, in accordance with (1.6)). Here, the velocities of the suspension and the bob are coordinated so that it appears that point O_1 momentarily stops. Such motion of the body corresponds to displacement of the image point along section c−0 of the phase-plane vector in Fig. 2b.

The body begins to move in the equilibrium regime at point 0 (Fig. 2b), when $\psi_\Delta \approx 0$, and the terms $-2\beta\cdot d\psi_\Delta/dt$ and $(R\cdot\eta/R(\pi + 1) \pm l)(d\psi^2/dt)^2$ in the right and left sides of Eq. (1.9) cancel out when $d\psi_\Delta/dt = V/R$. Uncontrollable coupling develops in this case, and the bob continues to move in accordance with phase-plane vector 0−a−0 in Fig. 2b (or over arcs 0−4−5 in Fig. 2a). This vector is described by Eq. (1.3) when $l < 0$. Further motion of the body corresponds to vector b−c−0 in Fig. 2b, in accordance with Eq. (1.9), when $l < 0$.

In the general case, the "upper" (0−a−b−c−0) and "lower" (0−a−b−c−0) sections of the phase-plane vector in Fig. 2b are asymmetric because the parameter l has different signs in the coefficients of differential equations (1.3) and (1.9). Only when $r(\pi + 1) \gg l$ (1.5) do the phase-plane vectors (0−a−b−c−0) and (0−a−b−c−0) become symmetrical, i.e., only then does the lower vector become the transformation of the upper vector (Fig. 2b).

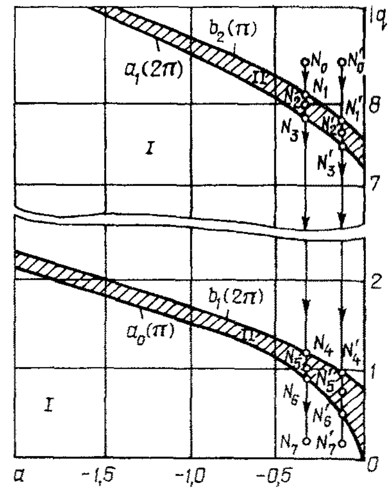

Fig. 3

The character of oscillation of the angle $\psi_\Delta(t)$ (or $\theta_\Delta(t)$) during the period T of motion of the suspension 0 (Fig. 2c) can be evaluated by a qualitative analysis of Eqs. (1.3) and (1.9) at $l > 0$ and analysis of the analogous equations for $l < 0$, as well as the phase-plane vectors in Fig. 2b. It follows from analysis of this figure that the period of vibration $T_1 \approx T_2$ of the body (point O_1) is roughly halved compared to the period T. For the case when $r(\pi + 1) \gg l$, the oscillations of $\psi_\Delta(t)$ become strictly symmetrical relative to the time axis $T_1 = T_2$ (Fig. 2c).

The process of motion of mass point O_1, of mass m, in an inverted pendulum with a vibrating suspension (Fig. 1b) represents a cyclic transition from equilibrium states at point 0 in noninertial coordinate system X_1OY to an equilibrium state at points b and b′ in inertial system XOY (Fig. 2b and c).

3. Let us now evaluate the conditions which ensure that the oscillations of $\psi_\Delta(t)$ (Fig. 2c) will be stable (steady). An analysis of Eq. (1.3), characterizing the motion of point O_1 and the image point on sections $0-a-$b and $0-a'-$b′ of the phase-plane vectors in Fig. 2b, shows that these vectors are stable in character. The motions of O_1 corresponding to sections $0-a-$b and $0-a'-$b′ in Fig. 2b are described by Mathieus; differential equations with the right side in the form of (1.9). The stability of such motions is conveniently studied by means of an Ains–Strett diagram [1, 4, 5, 10]. To do this, we need to change the left side of (1.9) to standard form by making the variable substitution $\psi_\Delta = ye^{-\beta t}$ (or $y = \psi_\Delta \cdot e^{\beta t}$). Then

$$\frac{d^2y}{dt^2} - (\beta^2 + \omega_0^2 \cdot \cos\omega_1 t) \cdot y = 0. \tag{3.1}$$

We make the following substitution in order to change over to dimensionless coefficients in Mathieu's equation (3.1): $\omega_1 t = 2\tau$; $t = \frac{2\tau}{\omega_1}$; $\frac{dy}{dt} = \frac{\omega_1}{2} \cdot \frac{dy}{d\tau}$ and $\frac{d^2y}{dt^2} = \left(\frac{\omega_1}{2}\right) \cdot \frac{d^2y}{d\tau^2}$. After inserting these values into (3.1) we obtain

$$\frac{d^2y}{d\tau^2} - \frac{4}{\omega_1^2} \cdot (\beta^2 - \omega_0^2 \cdot \cos 2\tau) \cdot y = 0. \tag{3.2}$$

Writing Eq. (3.2) with the dimensionless standard Mathieu equation in the form

$$\frac{d^2y}{d\tau^2} + (a - 2q \cdot \cos 2\tau)y = 0, \tag{3.3}$$

we see that

$$a = -\frac{4\beta^2}{\omega_1^2} = -\frac{R^2\eta^2}{[\,R(\pi+1) \pm l\,]^2}; \quad q = 2\left(\frac{\omega_0}{\omega_1}\right)^2 = \frac{2gR^2(\pi+1)}{V^2[R(\pi+1) \pm l\,]}. \tag{3.4}$$

Figure 3 shows part of the Ains–Strett diagram for the case being considered. Here, the axes are the dimensionless coefficients a and q (3.4). The part of the diagram shown corresponds only to negative values of the coefficient "a". The hatched regions II of the diagram are known [1, 4, 10] to be stable regions, while unhatched regions 1 characterize unstable vibrations.

The authors of [6, 7] used the method of equivalent transfer matrices to examine synchronization of the natural angular frequencies of working elements and parametric excitations arising in dynamic systems described by the Mathieu equation. It was shown that in the plane of the parameters a and q, the Ains–Strett diagram (see the part of the diagram in Fig. 3) determines the regions where a parametric excitation with the angular frequency ω_1 can (region II) and cannot (region I) be synchronized with the natural vibrations of a mechanical system with the angular frequency ω_0. The parameters that affect such synchronization in an inverted pendulum may be structural variables (R, l), the characteristic viscosity of the medium (η), or the velocity of the moving coordinate system (V).

Figure 3 shows two trajectories (N_0 N_1 N_2 N_3 N_4 N_5 N_6 N_7 and N_0' N_1' N_2' N_3' N_4' N_5' N_6' N_7') of the working point in an Ains–Strett diagram with an increase in V and constant values of the parameters R, l, and n. Synchronization is absent at the initial points N_0 and N_0', so that stable vibrations of an inverted pendulum with an oscillating suspension are impossible. The variables become stable when the working point on the Ains–Strett diagram falls within the narrow interval N_1-N_3 (or $N_1'-N_3'$) in the hatched region between boundary lines $b_2(\pi)$ and $a_1(2\pi)$. The vibrations with angular frequencies ω_0 and ω_1 are synchronized in this case, so that mass point O_1 undergoes complex vibrations of the form shown in Fig. 2c. To quantitatively evaluate the main harmonic components of these vibrations, we examine point N_2 inside the interval N_1-N_3 in Fig. 3. It is found that π- and 2π-periodic solutions of the Mathieu equation exist simultaneously at point N_2 [1, 4, 5, 10]. In fact, $2\cdot\omega_0^2/\omega_1^2 = 8$, from which $\omega_0/\omega_1 = 2$. On the other hand, $2\cdot(\omega_0/2)^2/\omega_1^2 = 8$, from which $\omega_0/\omega_1 = 4$. Thus, natural vibrations with the angular frequency ω_0 occur at point N_2, these vibrations simultaneously including components with the angular frequencies $2\omega_1$ and $4\omega_1$. It is not hard to see that vibrations of O_1 close to angular frequency $2\omega_1$ occur at point N_1, while vibrations of O_1 close to $4\omega_1$ occur at point N_3.

The results obtained in [1, 4, 10] confirm that stable vibrations of an inverted pendulum are possible when the frequency ratio ω_0/ω_1 is close to two. Using this relation, we find from (3.4) that

$$4 \approx \frac{gR^2(\pi+1)}{V^2\cdot[\,R(\pi+1)\pm l\,]}.$$

We can use this expression as a basis for choosing the design (R, l) and regime (V) parameters that will ensure realization of the above vibration synchronization region in a dynamic pendulum system with angular frequencies ω_0 and ω_1.

With a further increase in velocity V, the coefficient q decreases and synchronization becomes impossible. Synchronization becomes possible and the vibrations of point O_1 become stable when the working point falls within the narrow hatched region N_4-N_5 (or $N_4'-N_5'$) between boundary lines $b_1(2\pi)$ and $a_0(\pi)$.

It is evident from Fig. 3 that the lower the absolute value of the coefficient "a" (3.4), the higher the values of ω_1 = V/r at which stable vibrations of mass point O_1 of the inverted pendulum are possible. Stable vibrations of such a pendulum within this frequency range were studied experimentally by Academician P. L. Kapitsa and described in [2, 3].

4. In conclusion, we will restate the main features of the dynamic processes that take place in an inverted pendulum with a vibrating suspension.

1. For a specified periodic motion of the suspension (Fig. 1c), the body of the pendulum (mass point O_1 of mass m) undergoes complex oscillatory motion relative to the suspension point (Fig. 2c).

2. During such oscillation, the body goes from the equilibrium state characteristic of the inertial coordinate system to another equilibrium state characteristic of the noninertial coordinate system. There are two alternations between these equilibrium states in each cycle of motion of the suspension.

3. An analysis of the main equations of motion (1.3), (1.9) shows that steady vibrations of the bob of an inverted pendulum are excited and sustained as a result of internal positive nonlinear (quadratic) feedback with regard to the angular velocities $d\psi_\Delta/dt$ (1.9) and $d\theta_\Delta/dt$ (1.3) of the pendulum in noninertial and inertial coordinate systems. The character of motion of the bob during its vibration is determined by the changes which occur in this feedback.

4. An inverted pendulum can be used as a generator of steady mechanical vibrations (Fig. 2) when the suspension vibrates within certain frequency ranges.

REFERENCES

1. A. N. Maryuta, "Analysis of the motions of mechanical systems with frictional interaction," Prikl. Mekh., **25**, No. 10, 84-96 (1989).
2. A. N. Maryuta, "Mechanics of energy transfer by friction during rolling," ibid., **27**, No. 1, 136-144 (1991).
3. A. N. Maryuta, Theory of Modeling of Vibrations of the Working Elements of Mechanisms and Its Applications [in Russian], Izd. Dnepropetr. Univ., Dnepropetrovsk (1991).
4. I. Prigozhin and I. Stengers, Orders and Chaos. New Dialogue Between Man and Nature [in Russian], Progress, Moscow (1986).
5. T. S. Strizhak, Methods of Studying Dynamic Systems of the Pendulum Type [in Russian], Nauka, Alma-Ata (1981).
6. T. S. Strizhak, Averaging Method in Mechanics Problems [in Russian], Vyssh. Shk., Kiev.