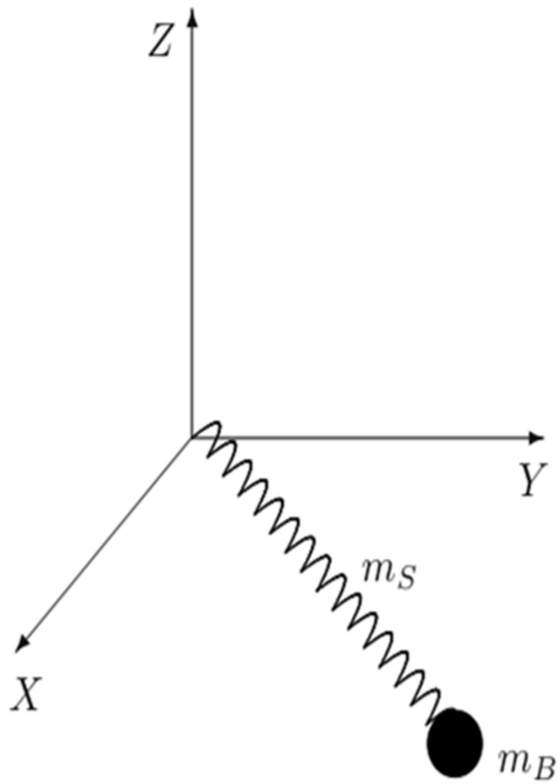
(Chaotic?) Heavy Spring Pendulum

2013 – Josh Villatoro and Hunter Ash

2011 - Jacob Stinnett and Logan Maingi

Advanced Lab II
Fall 2011

The System



Note: the pendulum component of the motion is modeled using the small angle approximation.

- Spring acts as pendulum arm
- Vertical oscillation unstable for arbitrarily small amplitudes when $$\frac{\Omega_P}{\Omega_S} = \frac{1}{2}$$

Ω_P : pendulum frequency

Ω_S : spring frequency

$$\Omega_P = \sqrt{\frac{g(m_B + \frac{m_S}{2})}{\ell(m_B + \frac{m_S}{3})}} \quad \Omega_S = \sqrt{\frac{k}{m_B + \frac{m_S}{3}}}$$

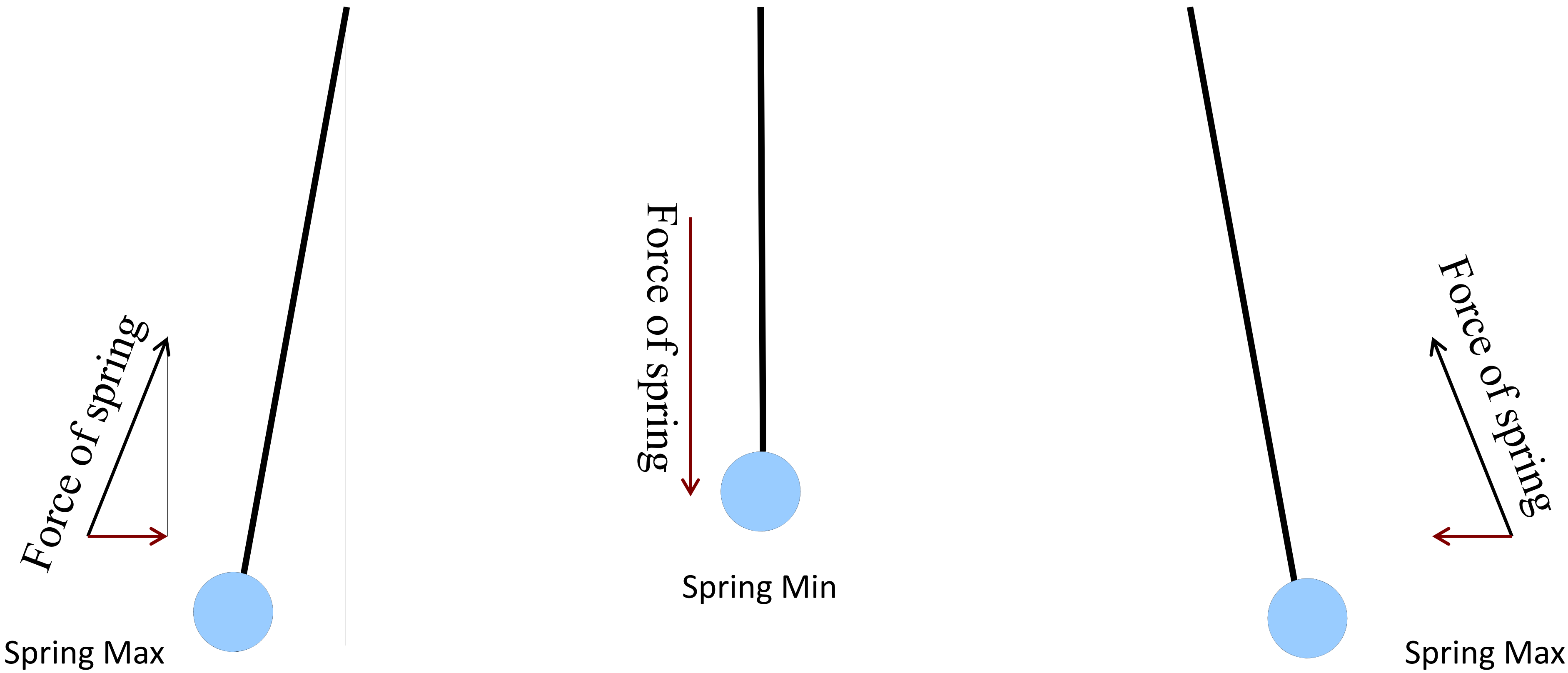
Setting $m_s = 0$ gives the frequencies for the light spring approximation.

Past Work

- Vitt and Gorelik studied small amplitude oscillations in the 2:1 resonance
- Cayton did numerical simulation and noted how the swinging plane was "apparently arbitrary"
- Rusbridge used a light bulb and film to record a swinging pendulum and similarly commented "some pendulums seem very bad for no obvious reasons (rotation of the swinging plane)"
- Lai studied slow varying amplitudes. He assumed that the finite spring mass should not affect the results
- Anicin and others studied the stability of small amplitude oscillations. They graphically determined instability of a particular amplitude as a function of mass
- Peter Lynch and others published the paper "The CO2 Molecule as a Quantum Realization of the 1:1:2 Resonant Swing-Spring with Monodromy"

From Pokorny's Paper

An Intuitive Explanation of the Instability at 2:1 Resonance



- At left position, spring force has a component rightward, and vice versa.
- At 2:1 spring to pendulum frequency resonance, spring phase is same for all angular maxima.
- Pendulum frequency angular dependence becomes significant for large oscillations (small angle approximation breaks down), taking the system out of resonance, which reduces horizontal amplitude, which brings the system back into resonance, which increases horizontal amplitude. This cycle continues as long as the system has energy.

A Brief Analysis (see end of ppt for more detailed analysis)

With Lagrangian mechanics, we can easily find the equations of motion.

$$E_{pot} = \left(m_B + \frac{m_S}{2}\right) gZ + \frac{k}{2}\left(\sqrt{X^2+Y^2+Z^2} - \ell_0\right)^2$$

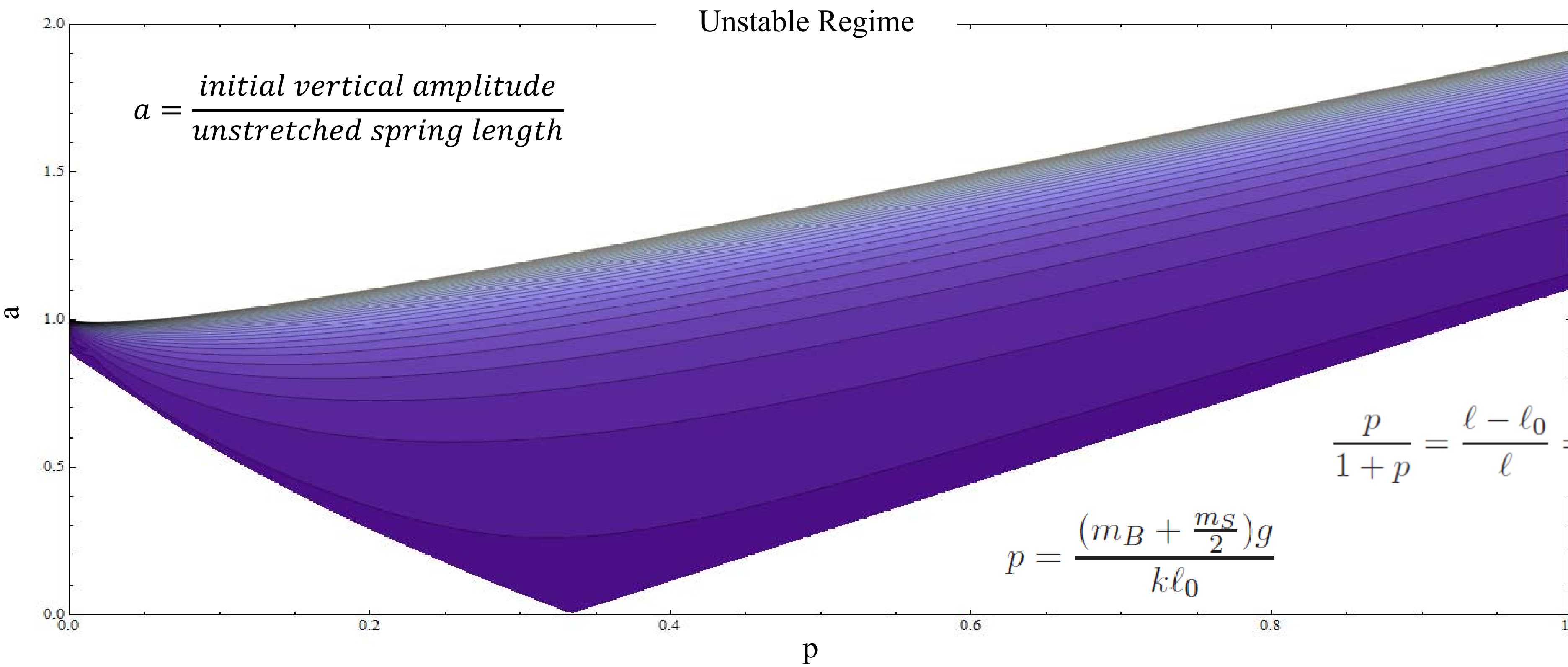
$$E_{kin} = (m_B + \frac{m_S}{3})\frac{d^2X}{dT^2} + (m_B + \frac{m_S}{3})\frac{d^2Y}{dT^2} + (m_B + \frac{m_S}{3})\frac{d^2Z}{dT^2}$$

However, this cannot be done analytically. Instead, we solve these numerically in Mathematica.

We then use the variational equation to study the instabilities.

$$\dot{M}(t) = f'(x(t)) \cdot M(t)$$

$$M(0) = \text{identity matrix}$$



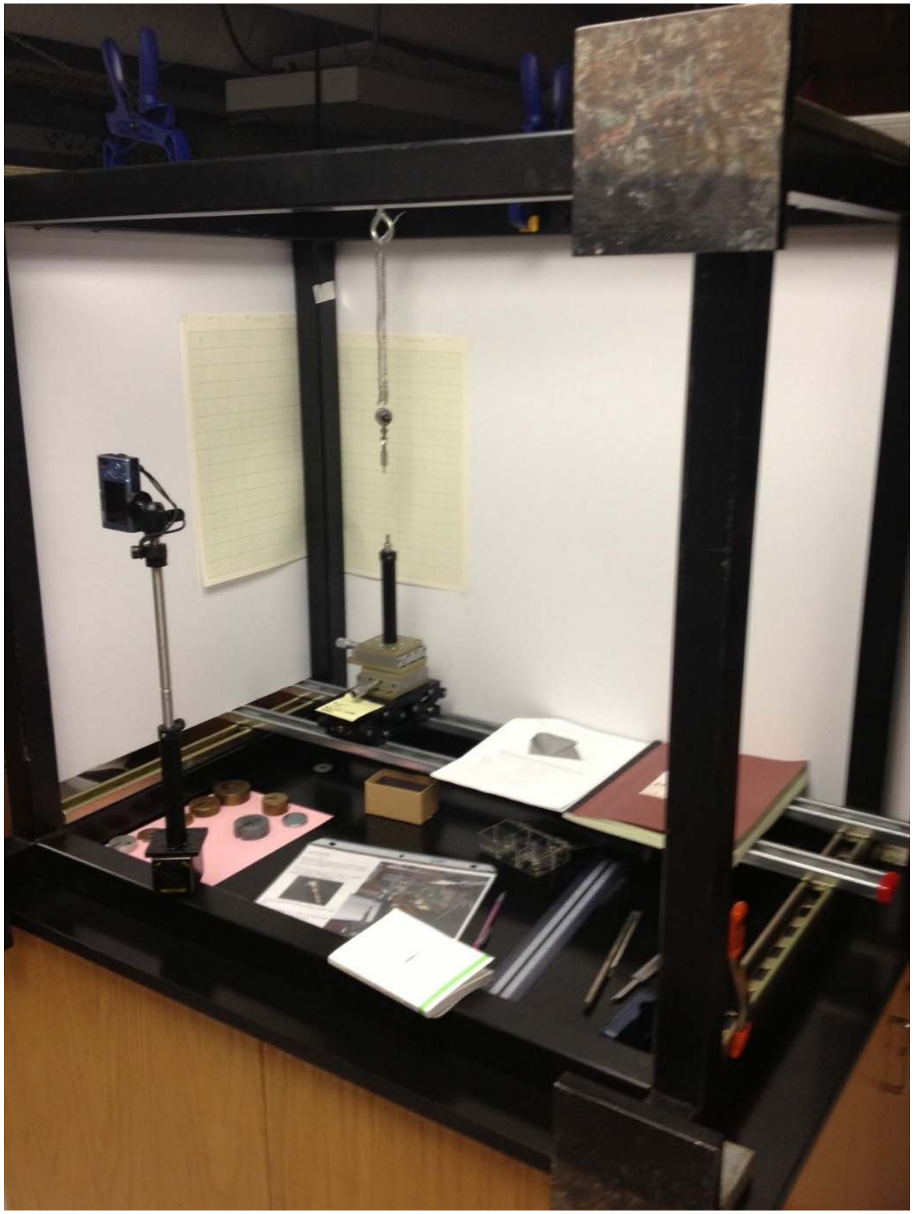
$$a = \frac{\textit{initial vertical amplitude}}{\textit{unstretched spring length}}$$

$$\frac{p}{1+p} = \frac{\ell - \ell_0}{\ell} = \left(\frac{\Omega_P}{\Omega_S}\right)^2$$

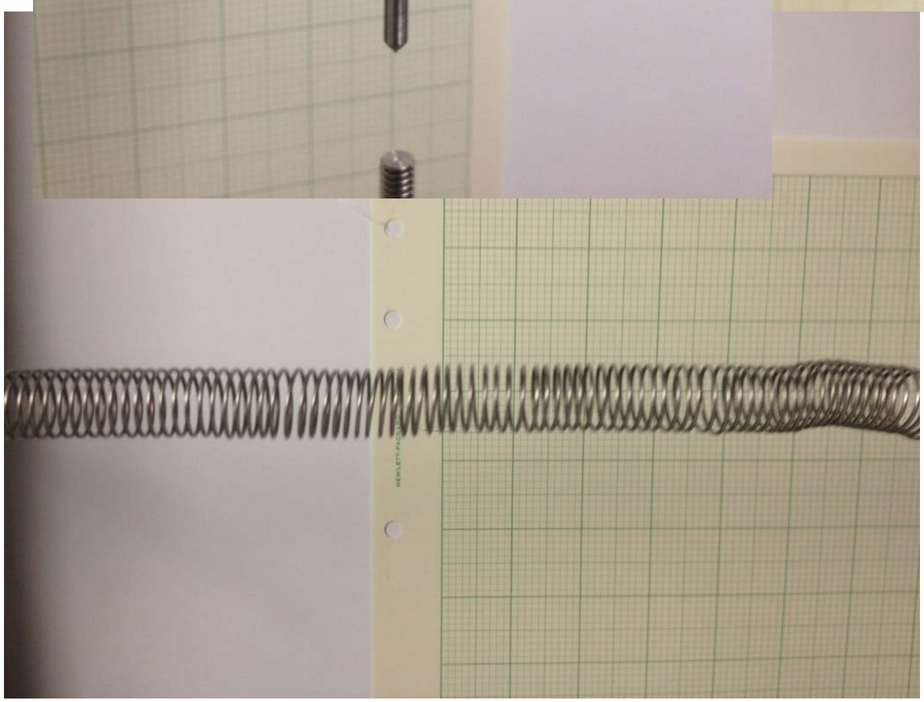
$$p = \frac{(m_B + \frac{m_S}{2})g}{k\ell_0}$$

The region shown above is the unstable regime. Contour lines denote the maximum eigenvalue of M(2π) (lighter colors->greater eigenvalues->greater instability). Note that the system is unstable for all a>0 at p=1/3, corresponding to the 2:1 spring frequency: pendulum frequency ratio, so the system is unstable at this resonance for arbitrarily small oscillations.

Lab Setup



Apparatus mounted on steel frame. Track made of unistrut bolted to frame. Release guide is on a fine-adjustable stage. Top surface of stage has been leveled. Graph paper on white background for measurement of horizontal displacement. Camera mounted approx. on level with pendulum bob.



Pendulum bob consists of bolt assembly, added masses secured with wingnut + lock washer, Pointer at end of bob that fits slightly into release guide to help stabilize release. That the bob is significantly spatially extended rather than pointlike is a possible source of error.

Note! The bob likes to tilt, but this can be corrected by rotating the added masses. More on next slide.

The spring is secured to the hook with a zip tie. The directional bias thus introduced is a possible source of error.

The spring is slightly wonky (see bottom left), a possible source of error.

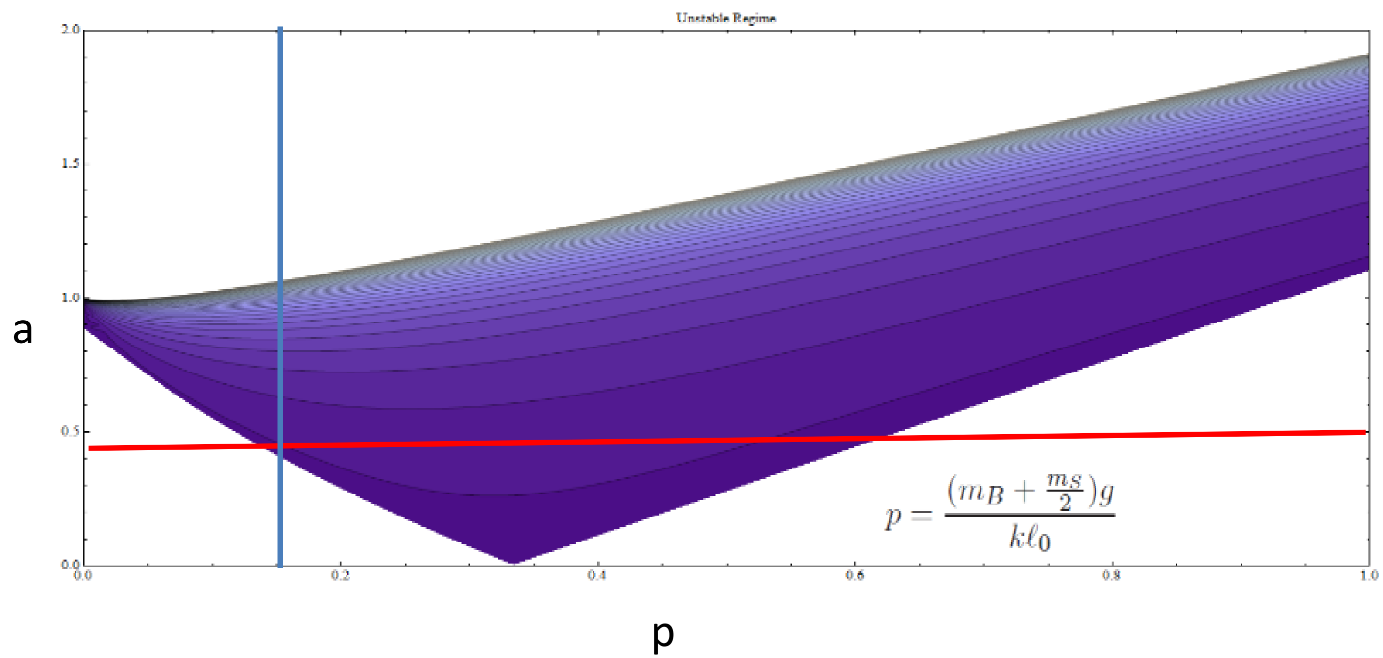
Also, the basement of Nielson is subject to constant high-frequency (relative to the spring) vibration due to the air handling unit = possible source of error.

The Amazing Josh Villatoro Bob Tilt Correction Algorithm™

- Note that the added masses are not circularly symmetric. They have a notch removed to allow for easy attachment. Their centers of mass are somewhere along the axis through the notch, on the other side of the center.
- First, align all of the notches and rotate them until they are parallel with the direction of tilt of the bob. This will either make the tilt maximally better or maximally worse. Doesn’t matter which.
- Now, separate the weights into two equal-mass groups. Rotate their notches away from the tilt axis by equal amounts. This draws the center of mass in towards the bolt. Continue rotating until minimum tilt is achieved. This is the minimum possible tilt for that amount of mass. Past a certain total added mass, the tilt is completely correctable, up to the manual dexterity of the experimenter.

Data Explanation

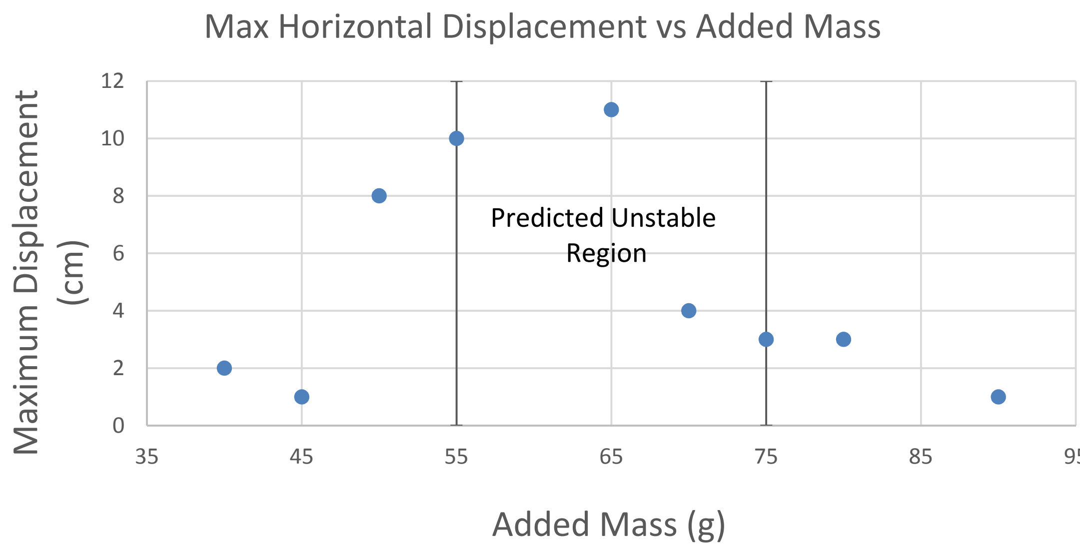
- We took two data series, corresponding to 1 horizontal and 1 vertical "cut" across the phase-space graph. The horizontal cut tested instability vs added mass, the vertical cut tested instability vs initial amplitude.



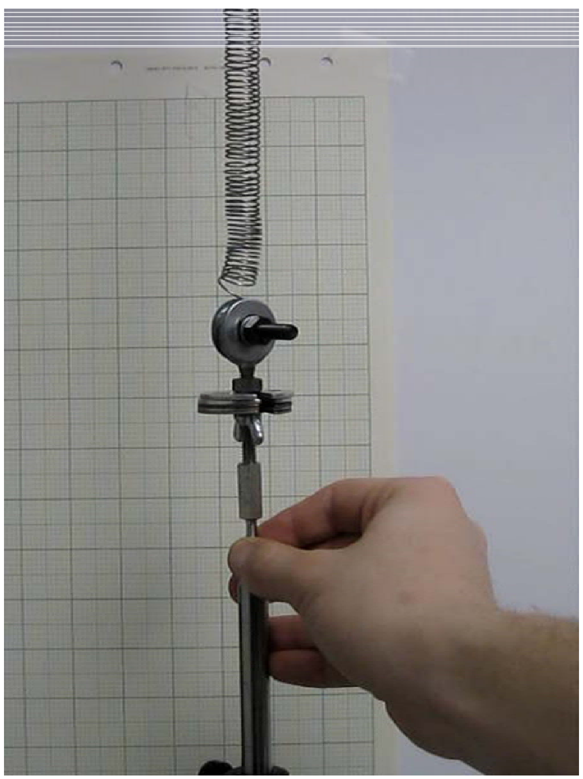
- For the horizontal cut, the predicted region of instability is an interval. For the vertical cut, there is a critical amplitude above which the system is unstable.

Data and Videos: Mass vs Instability

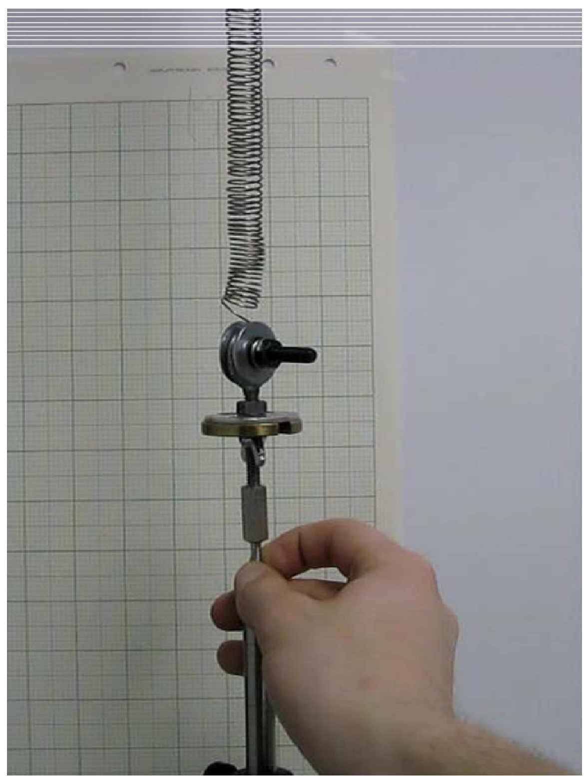
- Constant initial vertical amplitude = ½ inches = 1.27cm
- Max horizontal displacement tracked by eye by two observers looking along orthogonal axes, with eyes level with pendulum bob. Both observers note the max horizontal displacement they see. The reported max for a given trial is the greater of the two observed maxes.



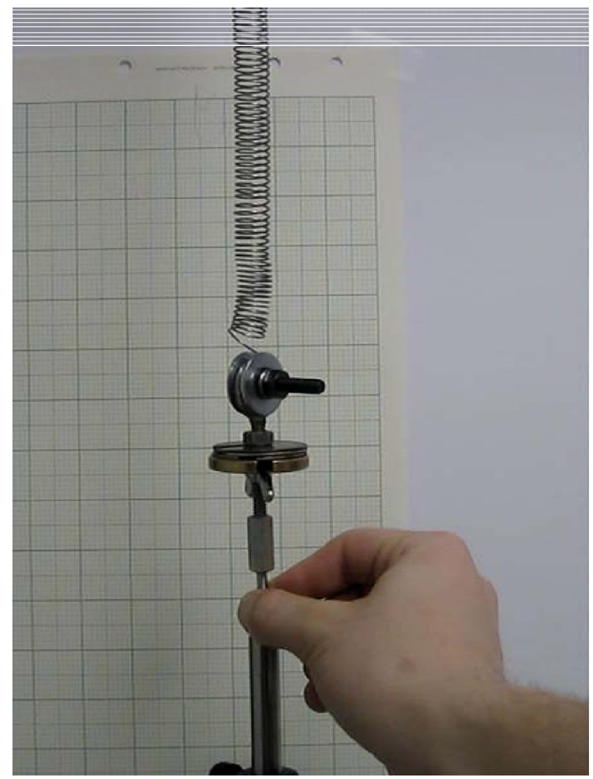
40g



55g

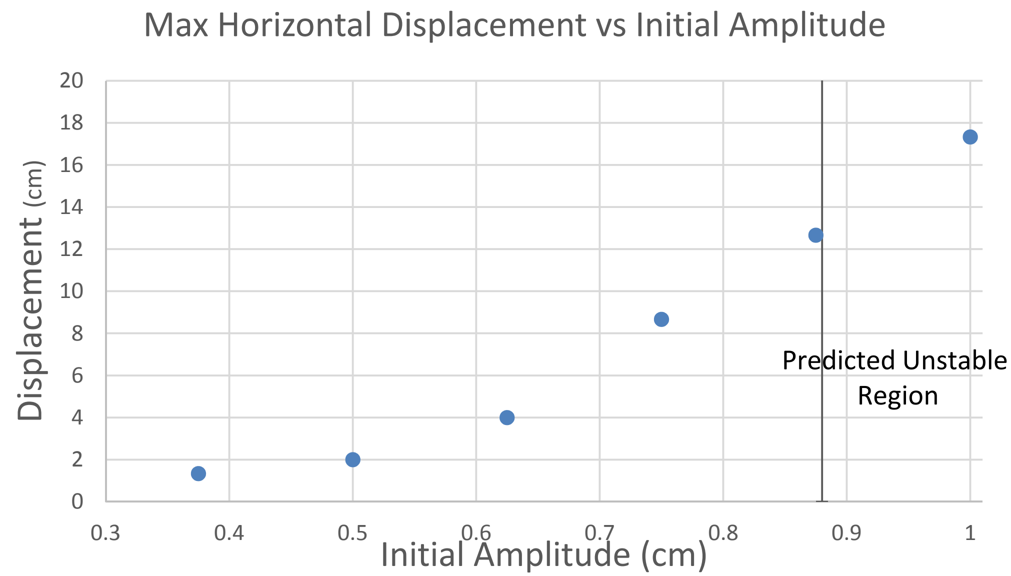


80g

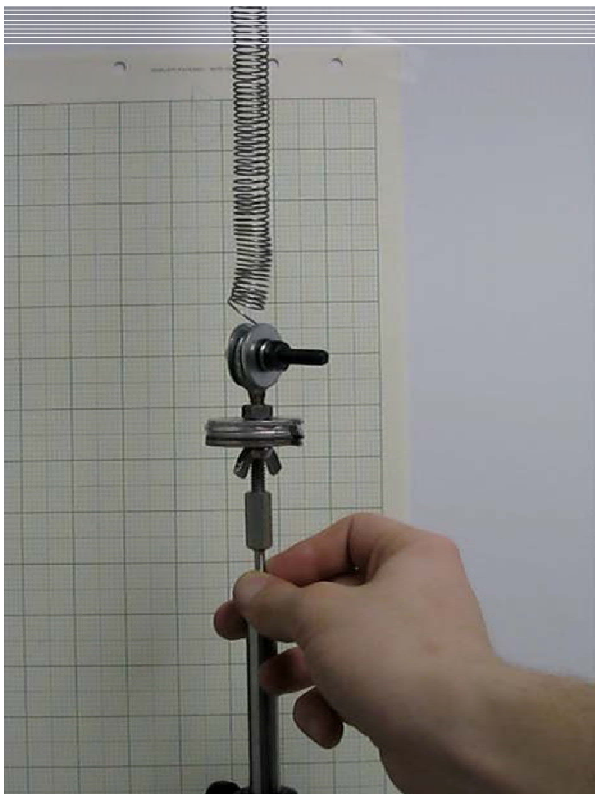


•Amplitude vs Instability

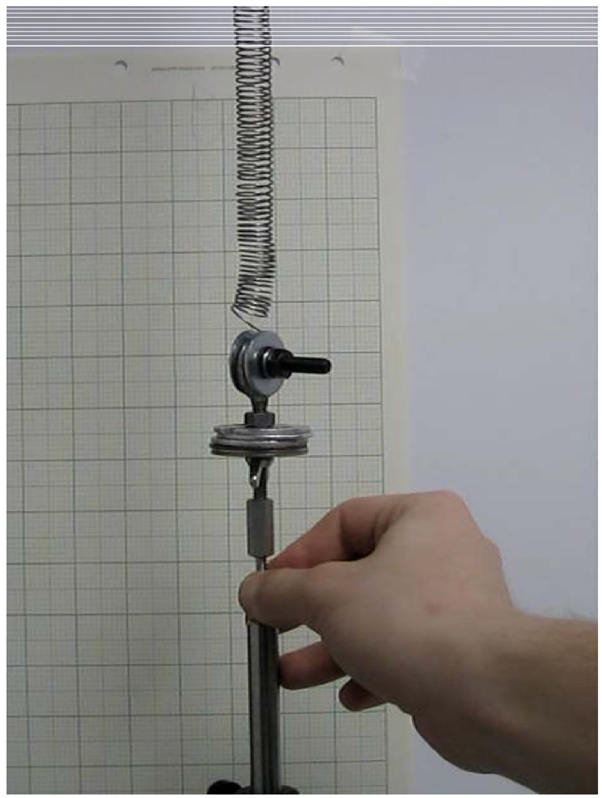
- Constant added mass of 45g



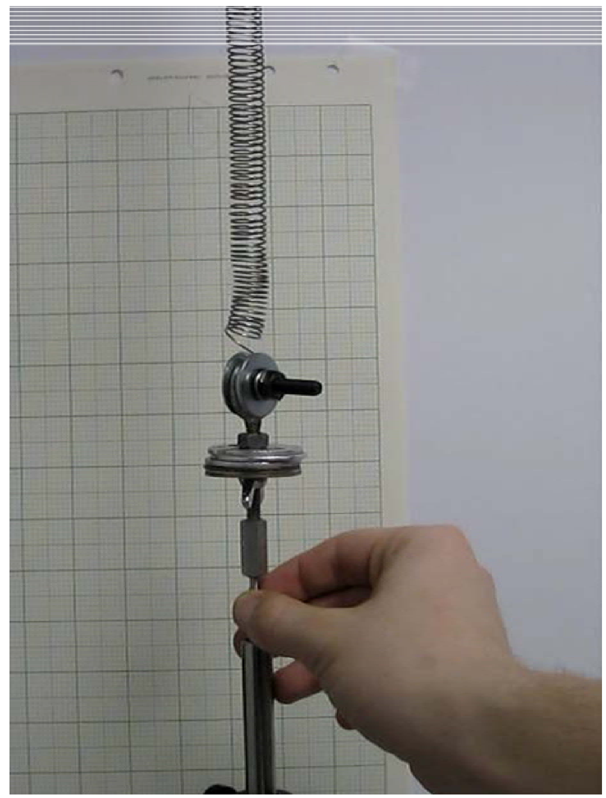
1.6cm|5/8 inch



1.9cm|3/4 inch



2.54cm|1 inch



Discussion of Results

Our results fit predictions roughly. Possible sources of error include:

- Zip tie attachment of spring to frame
- Shape of pendulum bob
- Wear in the spring between measurement of the spring constant and performance of the experiment
- Vibration of the building
- Rotation of swing plane causing max amplitude observations to be somewhat ambiguous.

Theoretical Analysis

Using Lagrangian mechanics, we can easily find the equations of motion.

$$E_{pot} = \left(m_B + \frac{m_S}{2}\right) gZ + \frac{k}{2}\left(\sqrt{X^2+Y^2+Z^2} - \ell_0\right)^2$$

$$E_{kin} = \left(m_B + \frac{m_S}{3}\right)\frac{d^2X}{dT^2} + \left(m_B + \frac{m_S}{3}\right)\frac{d^2Y}{dT^2} + \left(m_B + \frac{m_S}{3}\right)\frac{d^2Z}{dT^2}$$

$$\left(m_B + \frac{m_S}{3}\right)\frac{d^2X}{dT^2} = k\left(\frac{\ell_0}{\sqrt{X^2+Y^2+Z^2}} - 1\right)X,$$

$$\left(m_B + \frac{m_S}{3}\right)\frac{d^2Y}{dT^2} = k\left(\frac{\ell_0}{\sqrt{X^2+Y^2+Z^2}} - 1\right)Y,$$

$$\left(m_B + \frac{m_S}{3}\right)\frac{d^2Z}{dT^2} = k\left(\frac{\ell_0}{\sqrt{X^2+Y^2+Z^2}} - 1\right)Z - \left(m_B + \frac{m_S}{2}\right)$$

Making variables dimensionless and grouping the system constants into 1 parameter:

$$x = \frac{X}{\ell_0},\ y = \frac{Y}{\ell_0},\ z = \frac{Z}{\ell_0},\ t = T\sqrt{\frac{k}{m_B + \frac{m_S}{3}}}$$

$$p = \frac{(m_B + \frac{m_S}{2})g}{k\ell_0}.$$

$$\ddot{x} = \left(\frac{1}{\sqrt{x^2+y^2+z^2}} - 1\right)x,$$

$$\ddot{y} = \left(\frac{1}{\sqrt{x^2+y^2+z^2}} - 1\right)y,$$

$$\ddot{z} = \left(\frac{1}{\sqrt{x^2+y^2+z^2}} - 1\right)z - p,$$

Theoretical Analysis

We can then convert the system into six first order differential equations.

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (x, \dot{x}, y, \dot{y}, z, \dot{z})$$

$$\begin{aligned}
\dot{x_1} &= x_2, \\
\dot{x_2} &= \Big(\frac{1}{\sqrt{x_1^2 + x_3^2 + x_5^2}} - 1\Big)x_1, \\
\dot{x_3} &= x_4, \\
\dot{x_4} &= \Big(\frac{1}{\sqrt{x_1^2 + x_3^2 + x_5^2}} - 1\Big)x_3, \\
\dot{x_5} &= x_6, \\
\dot{x_6} &= \Big(\frac{1}{\sqrt{x_1^2 + x_3^2 + x_5^2}} - 1\Big)x_5 - p.
\end{aligned}$$

We want to study vertical oscillations about the bottom equilibrium, so the oscillations have the form

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 0, -1 - p + a \sin t, a \cos t)$$

The Variational Equation

The stability can then be analyzed by studying the variational equation

$$\left| \begin{array}{l} \dot{M}(t) = f'(x(t)) \cdot M(t) \\ M(0) = \text{identity matrix} \end{array} \right.$$

where $f'(x(t))$ is the Jacobian matrix (matrix of component-wise partial derivatives) evaluated at the lower equilibrium.

M(t) gives the time evolution of the system, i.e. x(t)=M(t)x(0).

Since the system is periodic about the equilibrium point, we only need to consider M(2 Pi).

The system is **stable** if the maximum norm of the eigenvalues of M(2π) is 1, i.e. they lie on or within the unit circle in the complex plane.

The system is **unstable** if the maximum norm of the eigenvalues of M(2 π) is greater than 1, i.e. at least one is outside the unit circle in the complex plane.

Note: For a full explanation of the variational equation, see Arrowsmith “Introduction to Dynamical Systems” or Arnol'd “Ordinary Differential Equations”.

Explanation of the Variational Approach

The system of equations may be written in matrix form:

$$\dot{x} = Ax$$

Perturb the known solution by a small amount δ(t) (which changes A to the perturbed matrix A') and take the first order term in series expansion:

$$e^{A't}\delta(0)$$

If the perturbation causes any of the exponentials in the matrix

$$e^{A't}$$

to have positive exponents, then the solution is unstable, since those exponentials will magnify any small offsets $\delta(0)$ in our starting conditions.

In our case, this instability condition is equivalent to

$$Max\{|\lambda_i|\} > 1$$

Where the λ_i are the eigenvalues of M(2π), M is the matrix from the variational equation.

The Variational Equation continued

- One case, the resonant case mentioned in the opening slide, is solved by hand in

ISSN 1560-3547, Regular and Chaotic Dynamics, 2008, Vol. 13, No. 3, pp. 155–165. © Pleiades Publishing, Ltd., 2008.

RESEARCH ARTICLES

Stability Condition for Vertical Oscillation of 3-dim Heavy Spring Elastic Pendulum

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- To investigate other parameter values, the system must be solved numerically. Pokorny's paper gives only the results; see Following_Pokorny's_paper.nb (Mathematica file) for the complete numerical analysis.

- Investigation of numerical plots yields instability condition

$$|a| \geq \left|(3p)^{\frac{2}{3}} - 1\right|$$

Sources

"Stability Condition for Vertical Oscillation of 3-dim Heavy Spring Elastic Pendulum":

P. Pokorny

Regular and Chaotic Dynamics, 2008

"Motion of the Sprung Pendulum"

M. G. Rusbridge

Department of Pure and Applied Physics, UMIST, 1979

Consultation with Nikola Petrov of the OU Math dept.