DEMONSTRATIONS

OF

DIFFRACTION AND SPATIAL FILTERING

BY

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Note to Teachers

Listed below are several items of interest concerning the diffraction and spatial filtering demonstrations:

(1) The diffraction demonstrations are of importance because they give the student an opportunity to relate the mathematics of diffraction theory to physical reality. The irradiance distribution of a particular diffraction pattern can be calculated and then compared with the observed irradiance pattern. The spation filtering demonstrations are important because the student can identify the Fraunhofer pattern of an object with the spatial-frequency spectrum of that object. Furthermore, he can manipulate the spectrum of the object and observe the effects on its image.

(2) These demonstrations should be suitable for presentation in optics courses in which diffraction is discussed, either at the graduate level or advanced undergraduate level. The time spent in setting up the demonstration will vary considerably depending on the equipment available, experience of the instructor, etc. The demonstration time itself can vary from 15 minutes to an hour and depends on the proficiency of the class, the desired outcomes, etc Student reactions at the Optical Sciences Center, University of Arizona, have been very good.

(3) The equipment costs may vary widely depending on quality, size, power output, etc. Suggested suppliers, with approximate costs, are given below:

(a) Laser. A laser with power output of 1 to 5 mw should be sufficient, and the cost should be between $300 and $1,000. The author used a Spectra-Physics Model 120 He-Ne Laser, supplied by Spectra-Physics, Inc., 1250 W. Middlefield Road, Mountain View, California 94040

(b) Lenses. Microscope objectives and cemented-doublet telescope objectives can be obtained for $5 to $25 depending on quality. Possible suppliers are Edmund Scientific Co., 555 Edscorp Building, Barrington, New Jersey 08007; or Rolyn Optics Co., 300 Rolyn Place, Arcadia, California 91006

(c) Accessories. Optical benches, lens holders, carriages, etc., may be purchased from Klinger Scientific Apparatus Corporation, 83-45 Parsons Boulevard, Jamaica, New York 11432; Ealing Beck, England, c/o The Ealing Corporation, Optics Division, 2225P Massachusetts Avenue, Cambridge, Massachusetts 02140. Costs will vary with quality.

(4) Photographs of diffraction patterns and spatially-filtered images can be found in the three references given at the end of the text, as well as in numerous other books on optics.

(5) Other information of use to teachers is included in the text.

DIFFRACTION AND SPATIAL FILTERING

I. Diffraction.

It is well known that there are many situations for which theories based on geometrical optics, or ray optics, do not adequately describe the behavior of light. To illustrate, let us consider the experiment depicted in Fig. 1; an aperture is illuminated by a collimated beam of light, and the irradiance patterns at various planes to the right of the aperture are observed. Ray theory predicts that each of these patterns will bear a geometrical resemblance to the aperture no matter how great the distance to the observing plane, but in fact they resemble the aperture for only a short distance. Beyond the so-called geometrical-optics region the light begins to spread out due to the phenomenon of diffraction, and the geometrical resemblance to the aperture is lost. The light patterns in this region, which is called the Fresnel region, undergo relatively rapid and dramatic changes as the distance to the observing plane is increased. Finally, a point will be reached beyond which only the size of the patterns, but not their shape, changes with increasing distance. At this point the Fraunhofer region, or far field, has been reached. In the sections to follow, mathematical descriptions and photographs of various diffraction pattern will be presented.

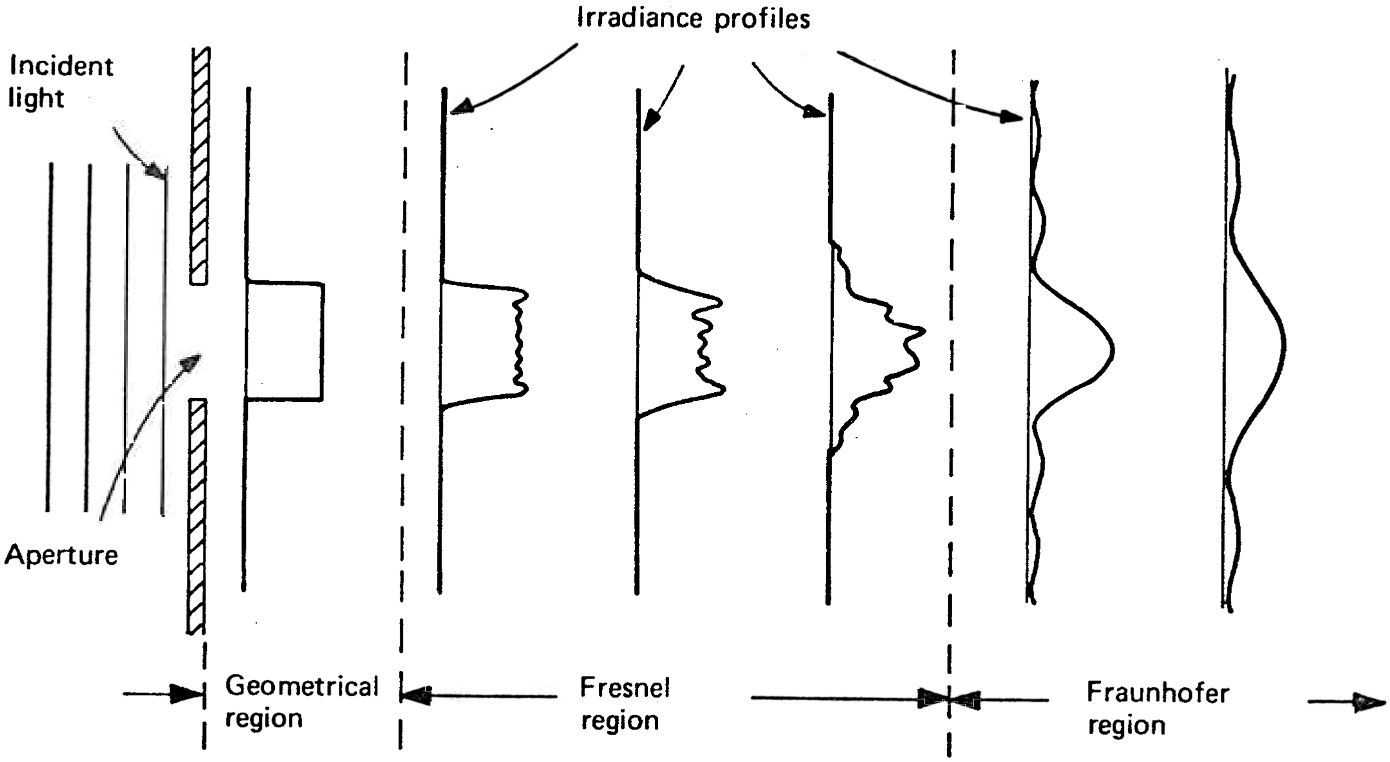


Fig. 1. Profiles of the irradiance patterns at various distances from a diffracting aperture.

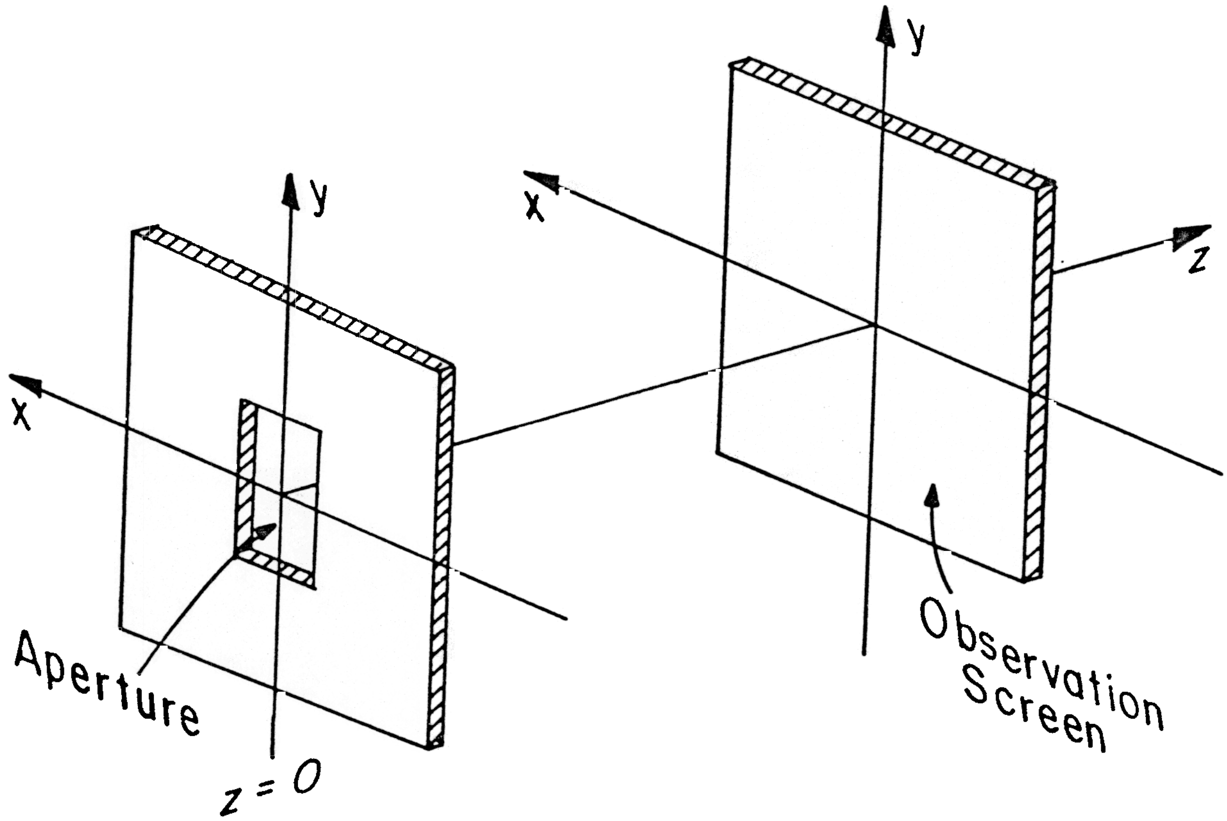


Fig. 2. Coordinate system used in the mathematical development.

A. Mathematical description of diffraction.

Under certain assumptions, the light distribution in either the Fresnel or Fraunhofer regions can be represented quite compactly by a general mathematical expression, which we shall present below. Our discussions will be based on the coordinate system of Fig. 2, and we make the following assumptions:

1. The incident light consists of normally-incident, quasi-monochromatic, linearly-polarized plane waves of unit amplitude.
2. The aperture is large compared to the wavelength of the incident light, but small with respect to the distance between the aperture and the observation screen
3. All observations are restricted to the paraxial region, i.e., to within approximately 15° of the z-axis

We shall describe the desired light distribution initially by its complex amplitude, a scalar quantity that corresponds to either the electric or magnetic field strength. If the complex amplitude transmittance of the aperture is designated $p(x,y)$, then the complex amplitude distribution just to the right of the aperture is given by

$$u(x,y,0) = p(x,y), \tag{1}$$

and, to within a multiplicative constant, the complex amplitude distribution in the observation plane may be written as (Ref. 1)

$$u(x,y,z) = \frac{1}{\lambda z} \iint\limits_{-\infty}^{\infty} p(\alpha,\beta) e^{j \frac{\pi}{\lambda z} (\alpha^2 + \;^2)} e^{-j \left(\frac{\alpha x}{\lambda z} + \frac{\beta y}{\lambda z}\right)} d\alpha d\beta. \quad (2)$$

Here λ is the wavelength of the light and z is the distance between the aperture and observation planes. This expression may be recognized as the Fourier transform of the product of the aperture function and a quadratic-phase factor. In general it is difficult to evaluate, but as we shall see, it can be evaluated with relative ease in certain special cases. The light pattern we observe visually is the irradiance, which is given by

$$E(x,y,z) = |u(x,y,z)|^2, \quad (3)$$

and we now calculate this quantity for a few special cases

Fraunhofer diffraction.

If the distance z is large enough with respect to the aperture size, then the quadratic-phase factor of (2) is approximately unity wherever the aperture function in nonzero. Thus $u(x,y,z)$ is simply the Fourier transform of the aperture function alone, i.e

$$u(x,y,z) = \frac{1}{\lambda z} P\left(\frac{x}{\lambda z}, \frac{y}{\lambda z}\right), \quad (4)$$

where $p(x,y)$ and $P(\xi,\eta)$ are a Fourier-transform pair, and ξ and η

are the spatial frequency variables associated with the x- and y-directions, respectively. Thus the irradiance of the observed pattern is

$$E(x,y,z) = \left(\frac{1}{\lambda z}\right)^2 \left|P\left(\frac{x}{\lambda z}, \frac{y}{\lambda z}\right)\right|^2 \tag{5}$$

Example 1: Consider the rectangular aperture of Fig. 2, with height d and width b. Then

$$p(x,y) = \begin{cases} 1, & |x| \leq \frac{b}{2} \text{ and } |y| \leq \frac{d}{2} \\ 0, & \text{elsewhere.} \end{cases} \tag{6}$$

Therefore,

$$P(\xi,\eta) = bd \ \mathrm{sinc}(b\xi, d\eta), \tag{7}$$

where

$$\mathrm{sinc}(\xi,\eta) = \frac{\sin(\pi\xi)}{\pi\xi} \ \frac{\sin(\pi\eta)}{\pi\eta} \tag{8}$$

and finally,

$$E(x,y,z) = \left(\frac{bd}{\lambda z}\right)^2 \mathrm{sinc}^2\left(\frac{bx}{\lambda z}, \frac{dy}{\lambda z}\right). \tag{9}$$

This expression is valid for $z \gg (b^2+d^2)/\lambda$. Note that the peak irradiance varies as the square of the aperture area and inversely as the square of the distance z, while the linear dimensions of the pattern vary directly as the distance z and inversely as the dimensions of the aperture. A normalized profile of this irradiance distribution along the x-axis is shown in Fig. 3, and except for scale, it is the same along the y-axis.

Example 2: For circularly-symmetric aperture functions, the Fourier transform becomes a Hankel transform of zero order. For the case of a clear circular aperture of diameter d, the transmittance function is given in polar coordinates by

$$p(r) = \begin{cases} 1, & r \le d/2 \\ 0, & r > d/2. \end{cases} \tag{10}$$

The Hankel transform of this is

$$P(\rho) = \frac{dJ_1(\pi d\rho)}{2\rho} \tag{11}$$

where J_1 is the first-order Bessel function of the first kind and ρ is the radial frequency variable in polar coordinates. Thus the irradiance is given by

$$E(r,z) = \left(\frac{\pi d^2}{4\lambda z}\right)^2 \left[\frac{2J_1\left(\frac{\pi dr}{\lambda z}\right)}{\frac{\pi dr}{\lambda z}}\right]^2, \tag{12}$$

which is the familiar Airy pattern. A normalized profile of this pattern is shown in Fig. 4, and its peak value and dimensions depend on d and z in a fashion similar to that for the rectangular aperture.

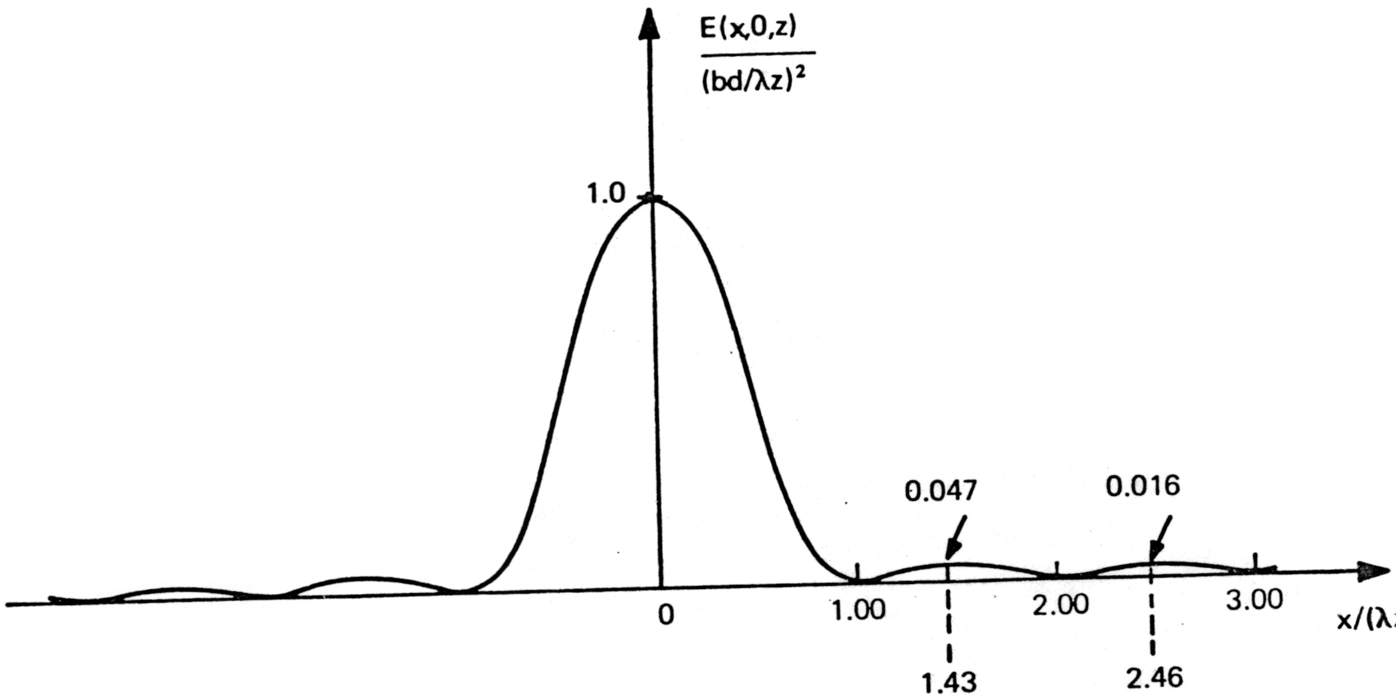


Fig. 3. Irradiance profile of the Fraunhofer diffraction pattern of a rectangular aperture with sides b and d.

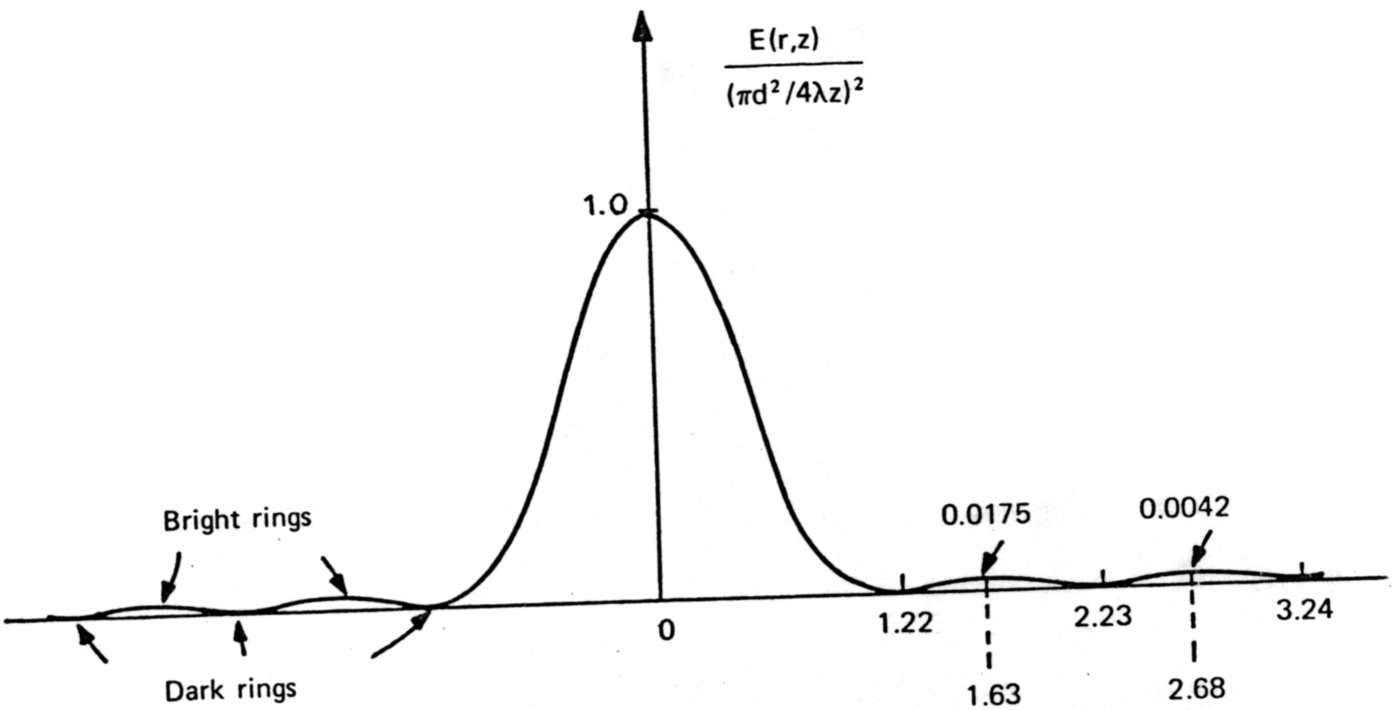


Fig. 4. Irradiance profile of the Fraunhofer diffraction pattern of a circular pupil of diameter d.

Fresnel diffraction.

For values of z too small to satisfy the Fraunhofer conditions, we have Fresnel diffraction. The quadratic-phase factor remains in the integrand of (2), and this integral can only be evaluated easily for a few special cases, one of which is the rectangular aperture case (see Ref. 1). The integral can also be evaluated to find the irradiance of the diffraction pattern of a circular aperture along the z-axis ($r = 0$).

Example 3: Consider a circular aperture of diameter d. For $r = 0$, the integral of (2) yields (after changing to polar coordinates)

$$u(0,z) = 2 \sin\left(\frac{\pi d^2}{8\lambda z}\right), \tag{13}$$

and the irradiance along the z-axis becomes

$$E(0,z) = 4 \sin^2\left(\frac{\pi d^2}{8\lambda z}\right). \tag{14}$$

For large values of z, the sine function becomes small and may be replaced by its argument. Thus we obtain

$$E(0,z) = \left(\frac{\pi d^2}{4\lambda z}\right)^2, \qquad z \gg \frac{d^2}{4\lambda}, \tag{15}$$

which exhibits the same behavior as the peak irradiance of (12). We see then that the Fresnel and Fraunhofer regions begin to merge at about $z = d^2/4\lambda$. If we now graph (14), we see that the irradiance on axis becomes zero at various points, which means there will be dark spots at these points (see Fig. 5).

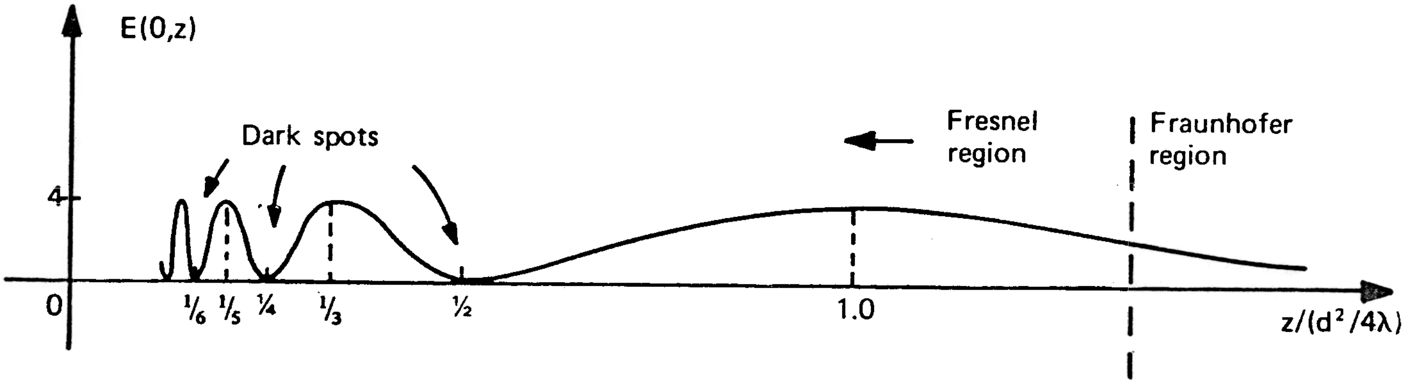


Fig. 5. Irradiance distribution along the z-axis in the Fresnel region of a circular aperture of diameter d.

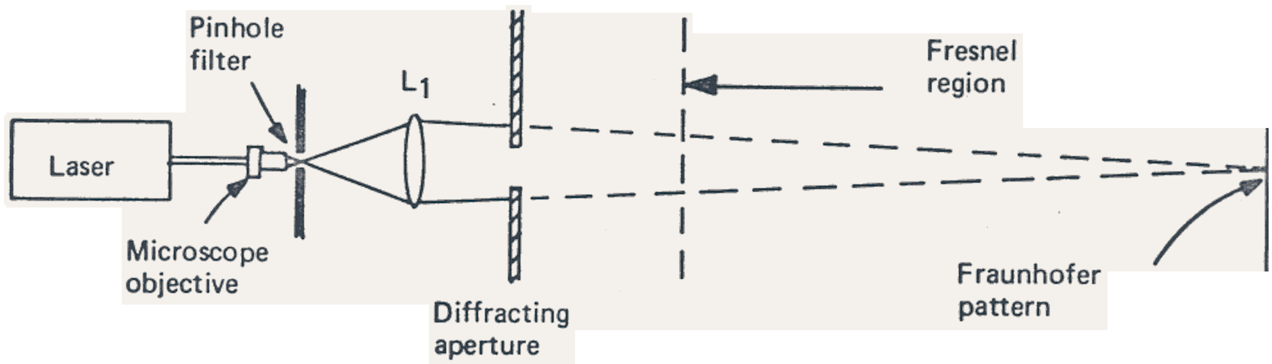


Fig. 6. Basic laboratory setup for displaying Fraunhofer diffraction patterns

TABLE I--REQUIRED EQUIPMENT

CW Laser -- (e.g., Spectra Physics Model 120, 5-mw He-Ne Laser.)
Assorted Diffracting Screens and Apertures -- (as suggested in text.)
2-meter Optical Bench and Accessories -- (lens holders, carriages, etc.)
12.5-μm Pinhole Filter -- (a 25-μm pinhole will also work with the 20X objective.)
10X and 20X microscope objectives.
L_1 -- (5-cm focal length coated doublet.)
L_2 -- (20-cm focal length coated doublet.)
L_3 -- (15-cm focal length coated doublet.)
Observation screen -- (large white card)

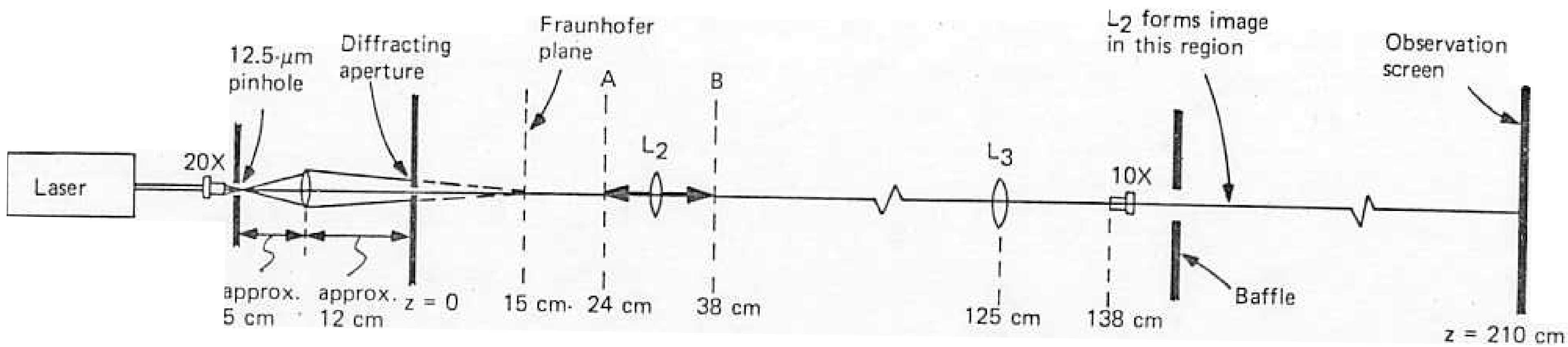


Fig. 7. Optical system for diffraction and spatial-filtering demonstrations.

B. Demonstration of diffraction.

For apertures with dimensions on the order of millimeters or centimeters, the distance z must be quite large to observe Fraunhofer diffraction. This would seem to make a demonstration of diffraction quite difficult to perform; however, such a demonstration is possible with the use of a positive "Fourier-transforming" lens and some auxiliary optics. The basic setup is depicted in Fig. 6. The Fraunhofer pattern of the diffracting aperture is observed in a plane conjugate to the pinhole, and the location of this pattern therefore depends on the focal length of L_1. (The pinhole is not absolutely necessary, but without it the observed pattern may contain some unwanted artifacts.) In addition, Fresnel diffraction can be viewed in the region between the diffracting aperture and the Fraunhofer plane. Many of the desired patterns will be too small to observe easily, but with a properly chosen reimaging system, magnified images of these patterns, and of the diffracting aperture as well, can be projected onto a screen and viewed with the unaided eye. A configuration used successfully by the author is shown schematically in Fig. 7, and it allows the demonstration to be set up in a reasonably small space (less than three meters), but other reimaging systems may also be used. (The description of the required equipment is listed in Table I.) With L_2 at position A, the aperture will be imaged onto the observation screen with a magnification of about

60. Actually, L_2 forms an image of the aperture somewhere to the right of L_3, and then L_3 and the 10X microscope objective produce the final image. As L_2 is moved to the right, the Fresnel region is imaged onto the screen, and finally, when L_2 reaches position B, the Fraunhofer pattern will be displayed with a magnification of approximately 75. (Note that the locations given in Fig. 7 are for the particular set of lenses listed, and different locations will be required for other lens combinations.) There may be some vignetting with this set up, but it usually will not be serious enough to ruin the demonstration. If necessary, however, more of the Fraunhofer pattern can be observed by moving the diffracting aperture closer to the Fraunhofer plane. A photograph of the system is shown in Fig. 8.

Procedures.

Once the system is properly set up and aligned, the following observations should be made:

1. Observe the images and diffraction patterns of several rectangular apertures with dimensions between 0.5 and 3.0 mm (see Fig. 9). Note the inverse dependence of the Fraunhofer pattern width on aperture width, and also note how the peak irradiance increases with aperture area.
2. Repeat Step 1 for several circular apertures with diameters between 0.5 and 3.0 mm (see Fig. 10)

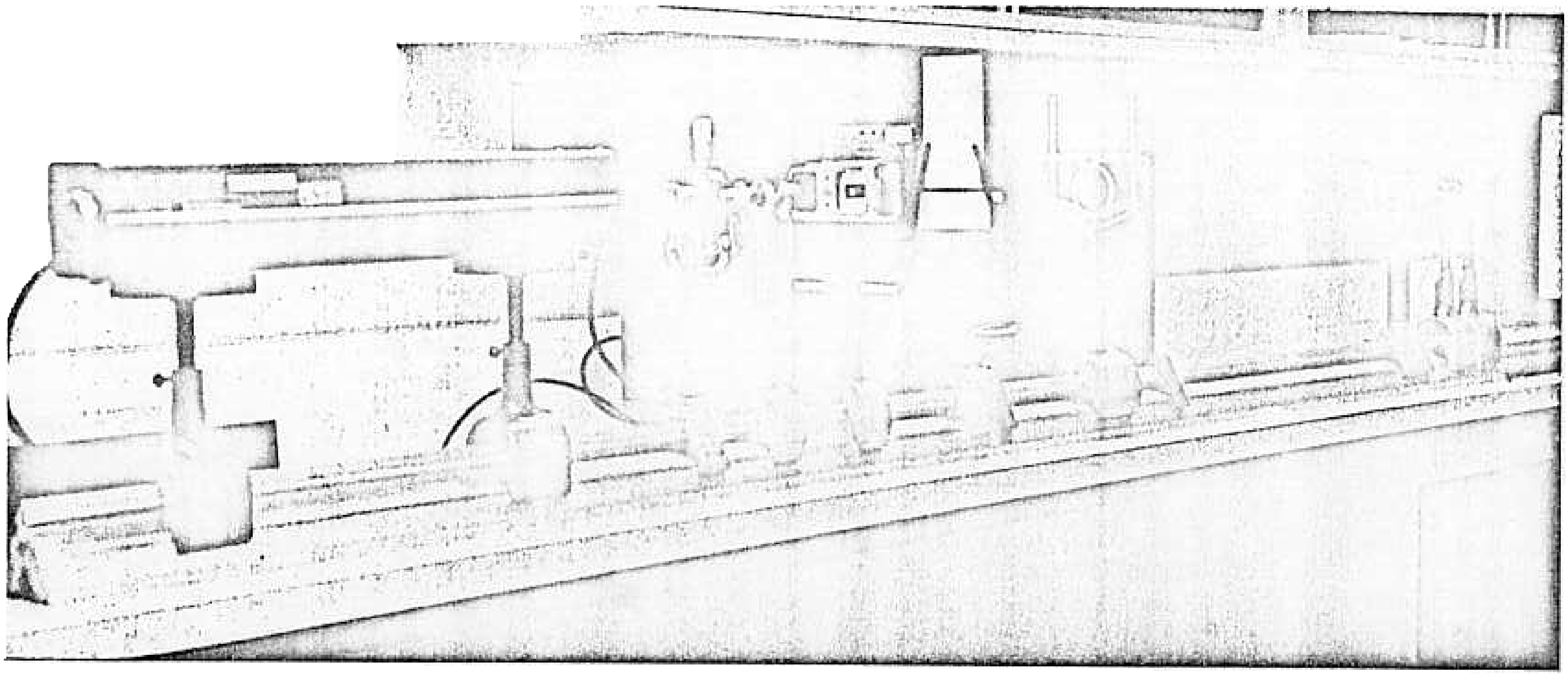
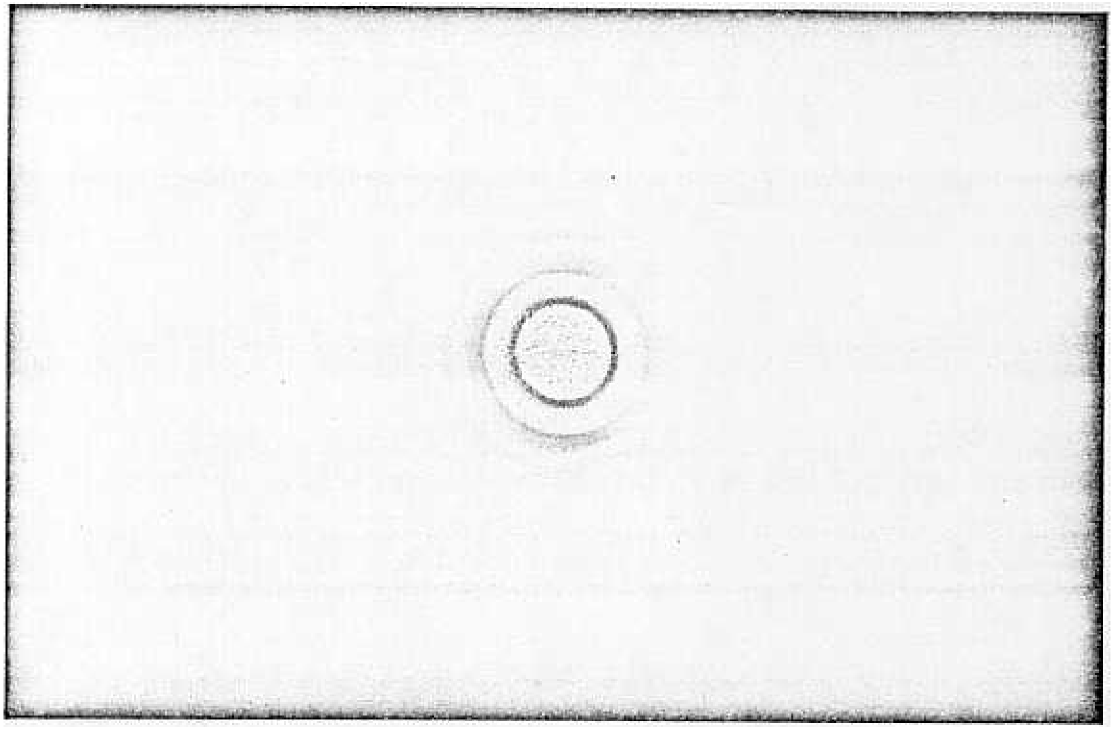
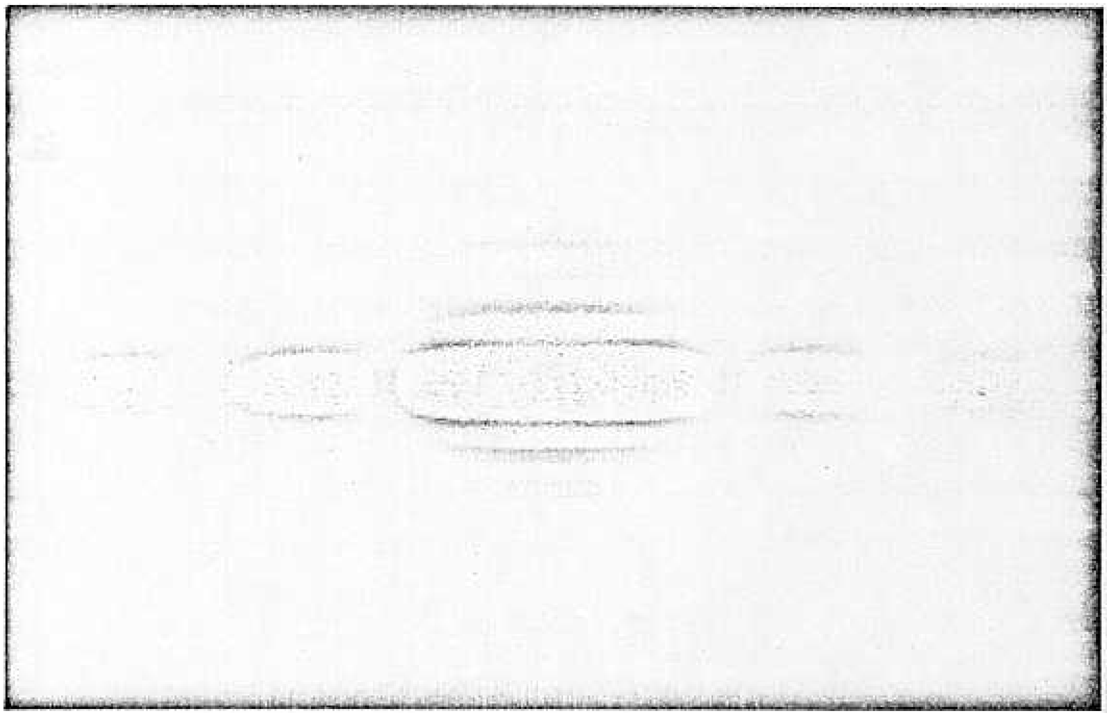


Fig. 8. Photograph of the system used to obtain diffraction patterns and images of Figs. 9-22.

Fig. 9. Fraunhofer diffraction pattern of rectangular aperture of

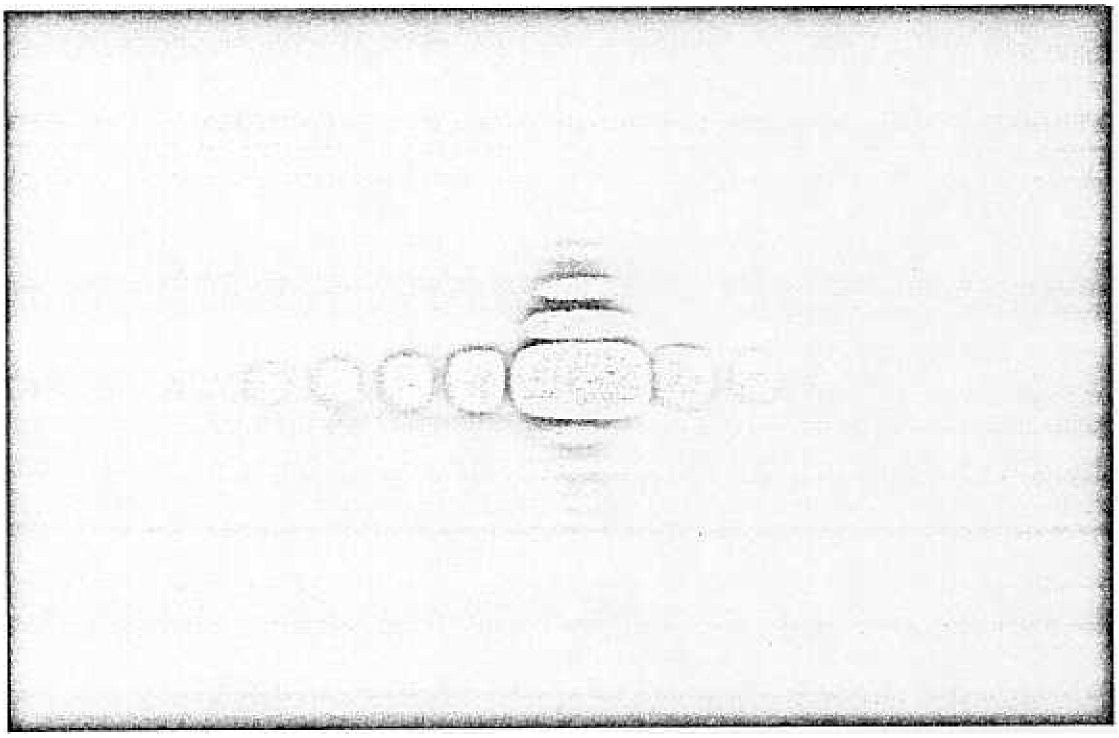


(a) height 3.0 mm and width 1.5 mm

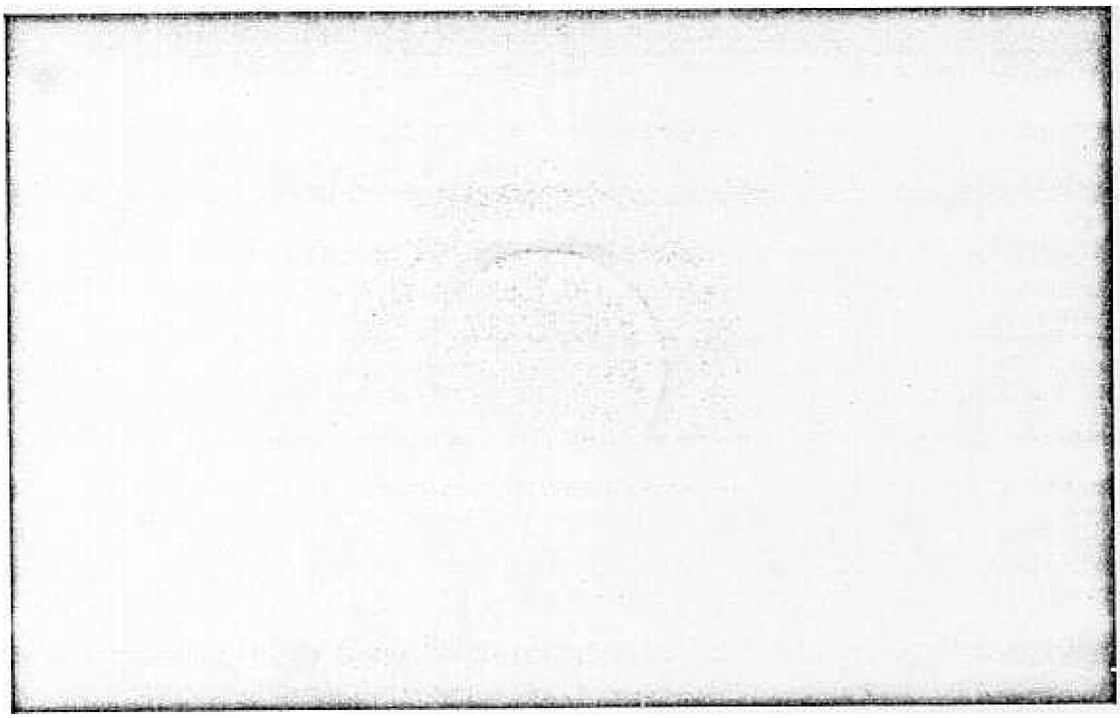


(b) height 3.0 mm and width 0.6 mm.

10. Fraunhofer diffraction pattern of circular aperture of



diameter 2.7 mm



diameter 1.0 mm.

3. Repeat Step 1 for a triangular aperture of approximately the same size, and note how light is diffracted in directions perpendicular to each side (see Fig. 11). Can you think of an aperture shape that will produce a diffraction pattern resembling a "five-pointed star?" (Note that star images obtained with reflecting telescopes have "points" due to diffraction by the spider assembly.)
4. Use a circular diffracting aperture and start with re-imaging lens L_2 at position A. Slowly move L_2 to the right, traversing the Fresnel region, and observe the irradiance on axis (see Fig. 13). Note how the irradiance becomes zero at certain points as predicted by (14). Repeat for apertures of different diameters, and note how the spacing of these minima changes with aperture size
5. Place a fine wire mesh with approximately 5 to 10 wires/mm (e.g., Buckbee Mears 250 Mesh Nickel) over one of the diffracting apertures, and observe how the Fraunhofer pattern now consists of an array of patterns, each of which is just the pattern associated with the diffracting aperture itself. Note that the scale of the array does not change as the diffracting aperture size is changed, but that the size of the

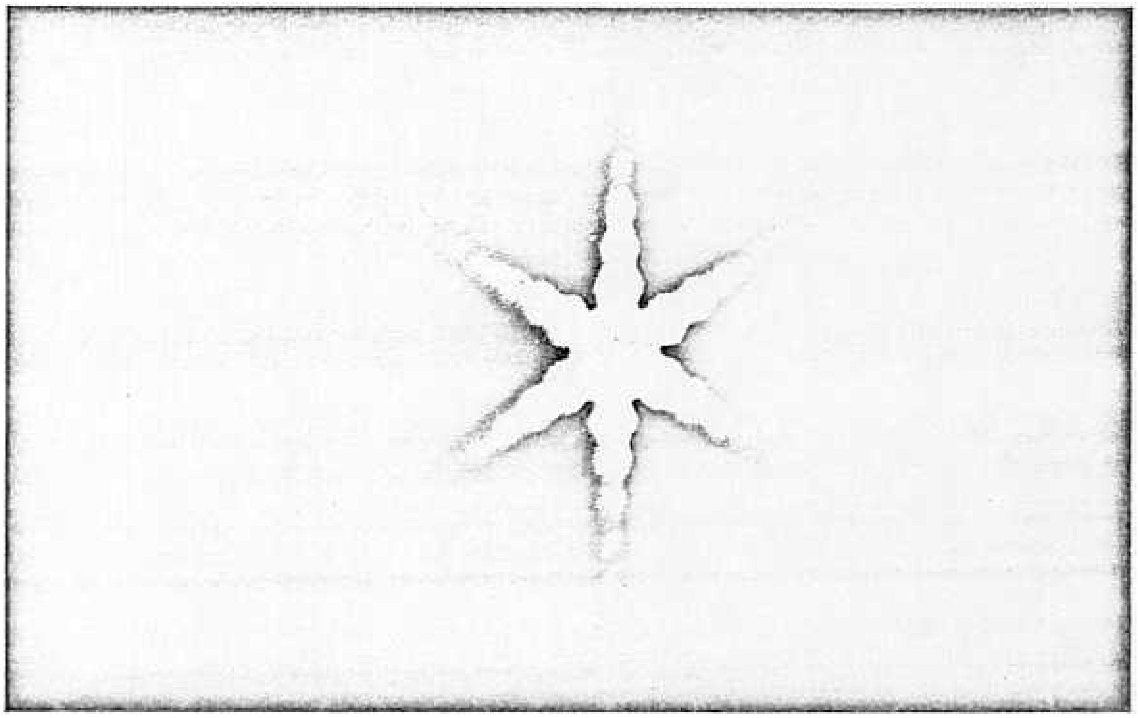


Fig. 11. Fraunhofer diffraction pattern of triangular aperture with sides of 1.7 mm.

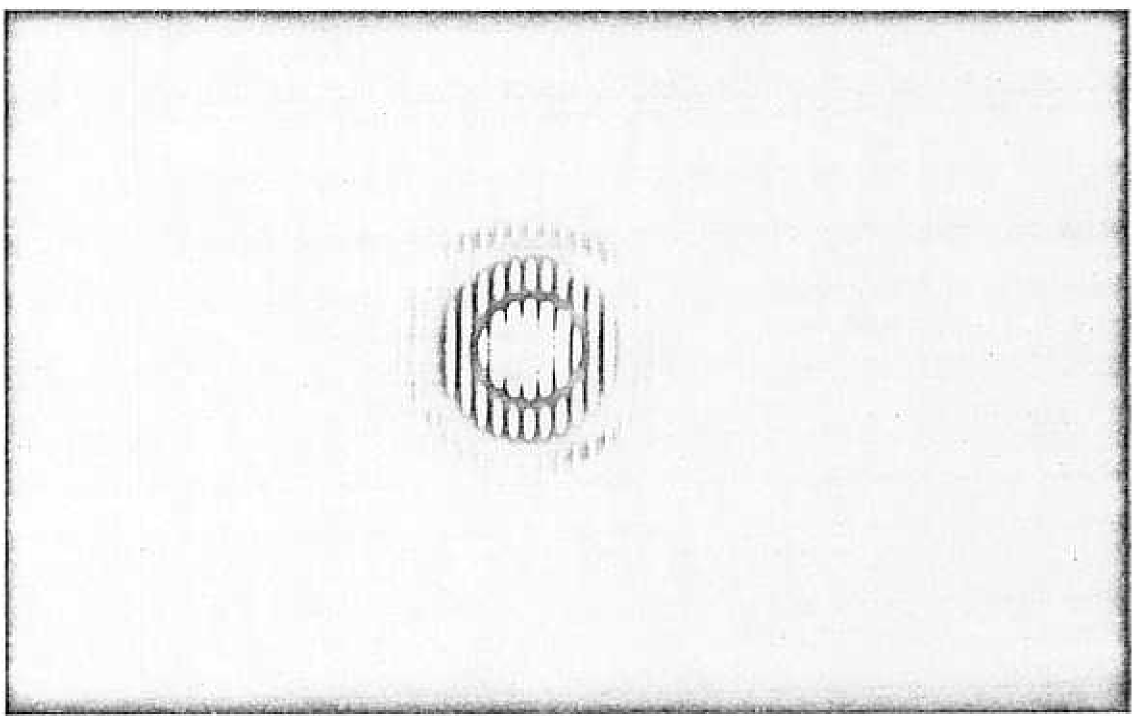


Fig. 12. Fraunhofer diffraction pattern of two circular apertures of 0.5 mm diameter and horizontal center-to-center separation of 2.25 mm.

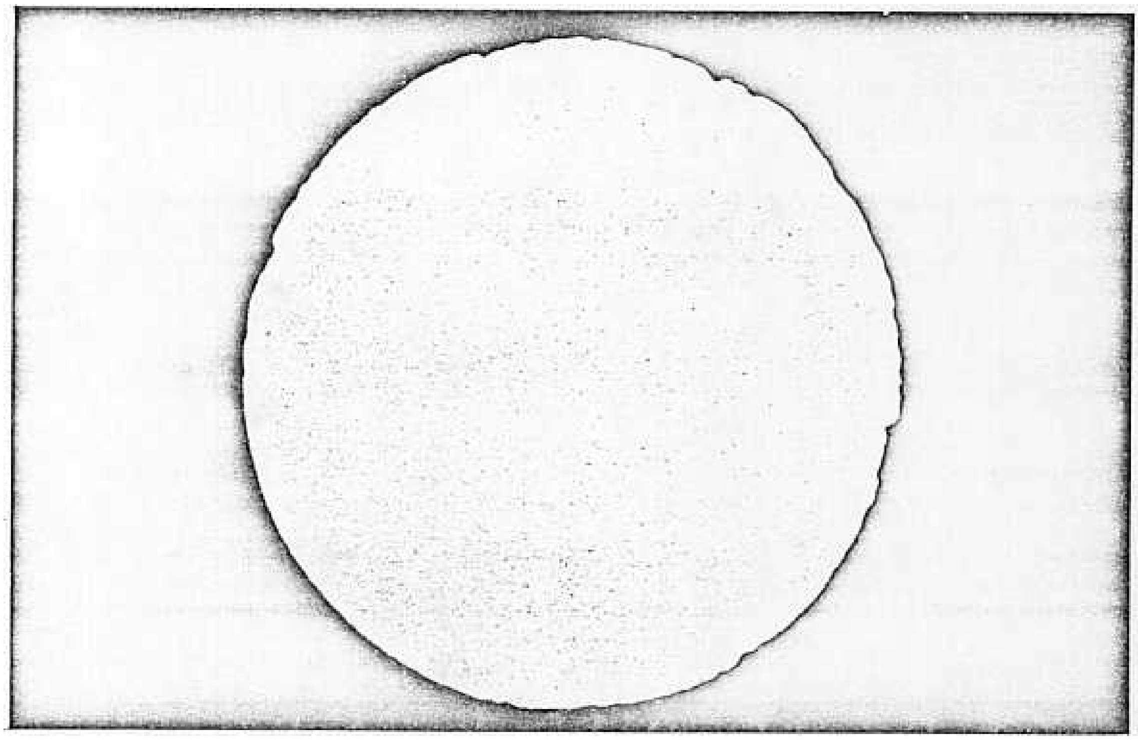


Fig. 13. (a) 1.7-mm diameter aperture

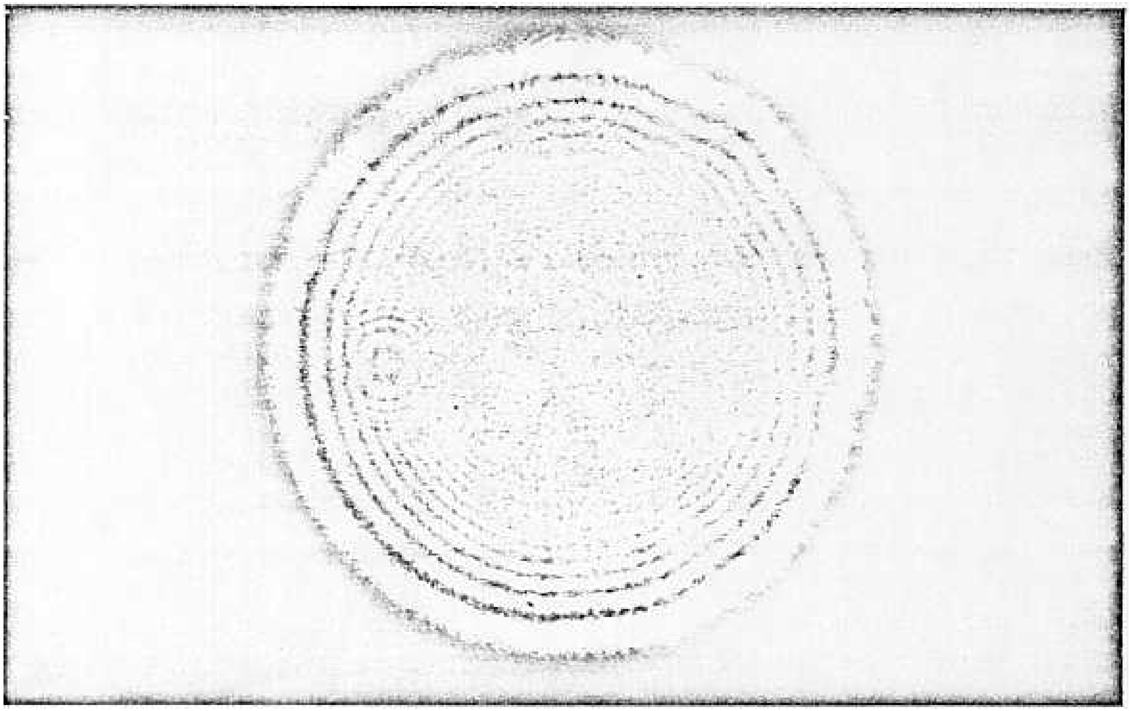


Fig. 13. (b) Fresnel diffraction patterns of this aperture at various distances from it

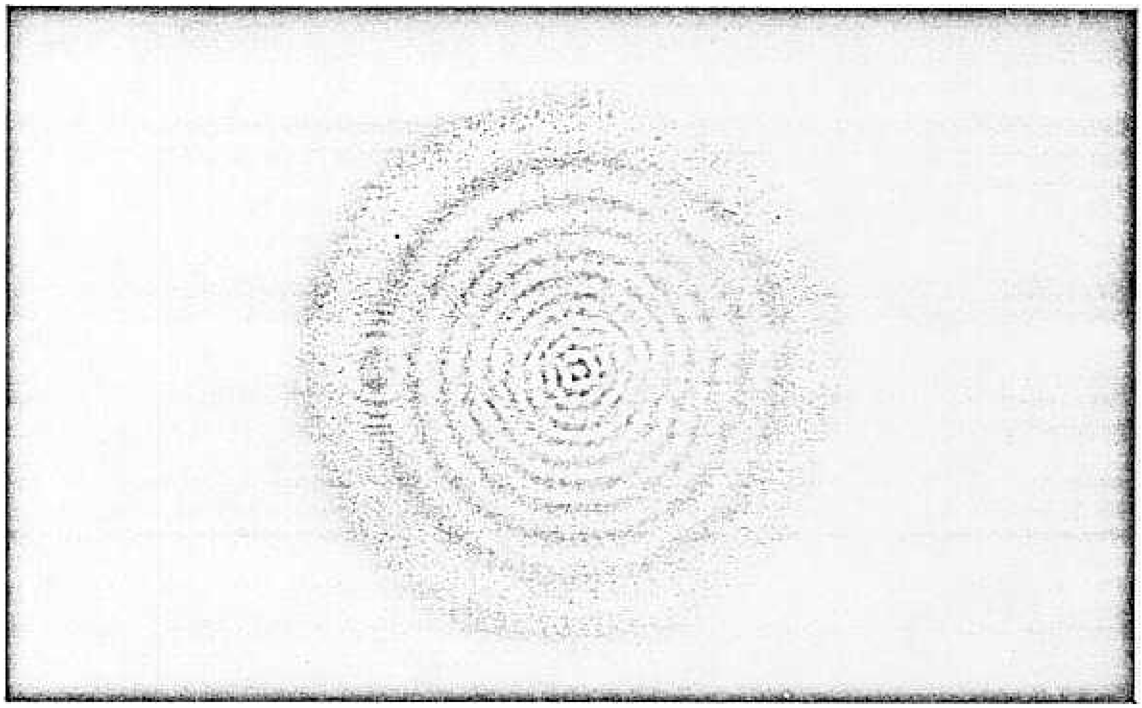


Fig. 13. (c) Fresnel diffraction patterns of this aperture at various distances from it

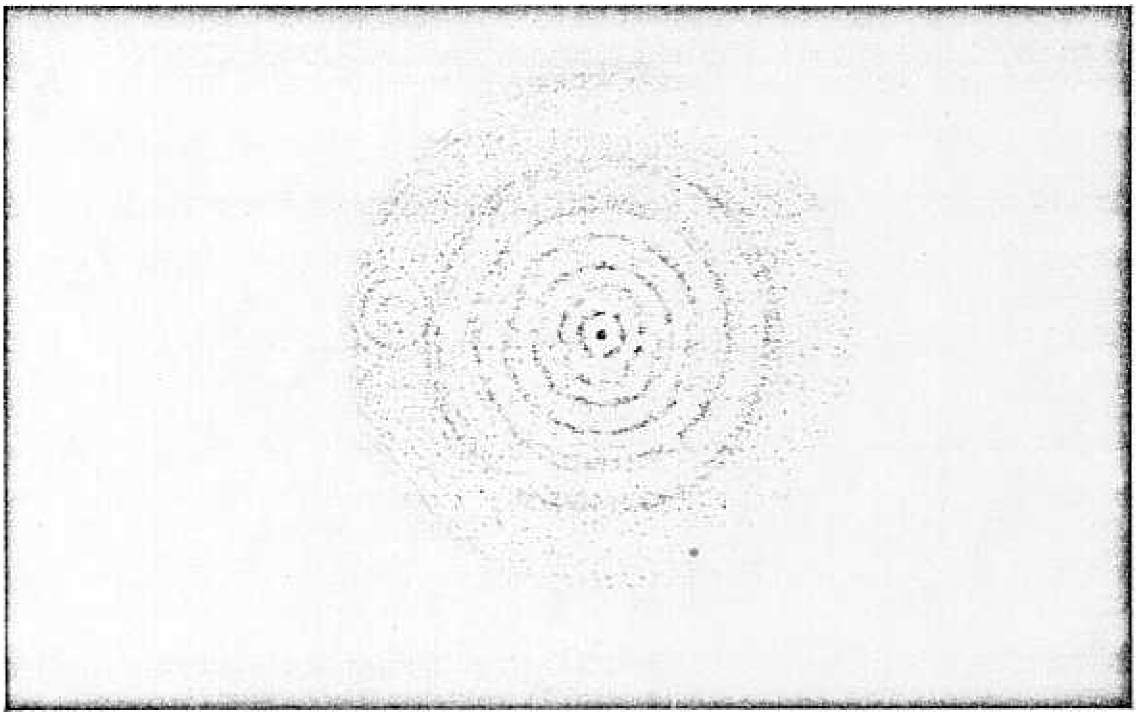


Fig. 13. (d) Fresnel diffraction patterns of this aperture at various distances from it

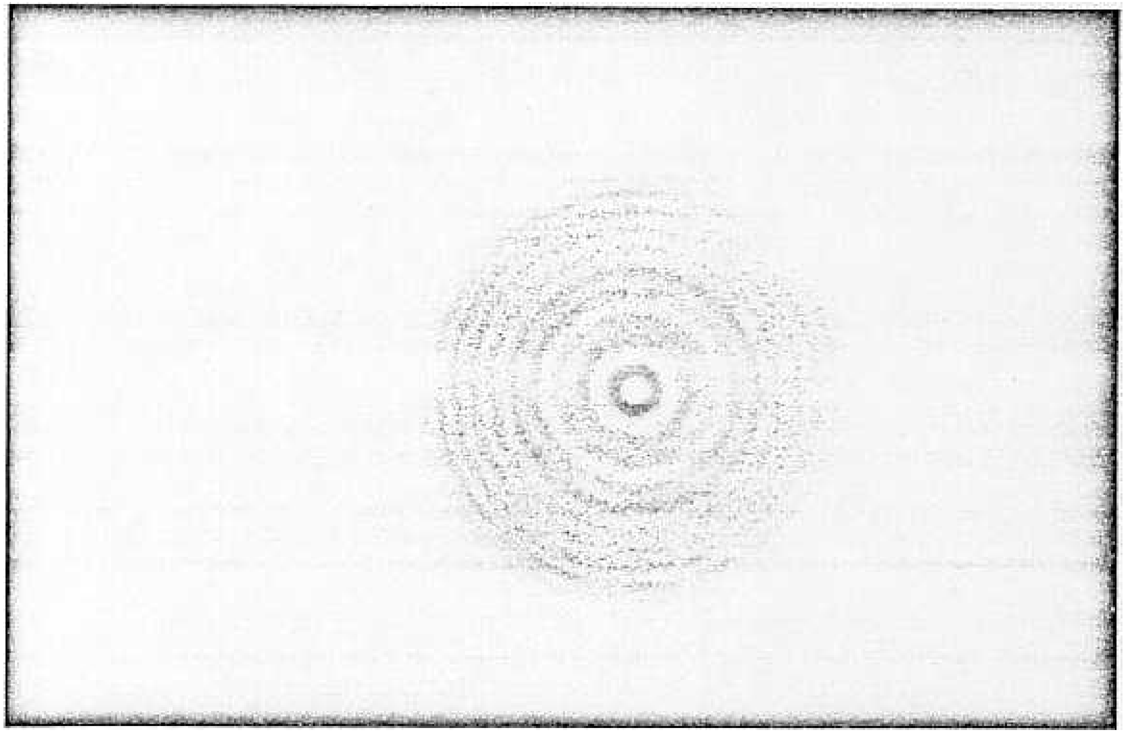


Fig. 13. (e) Fresnel diffraction patterns of this aperture at various distances from it

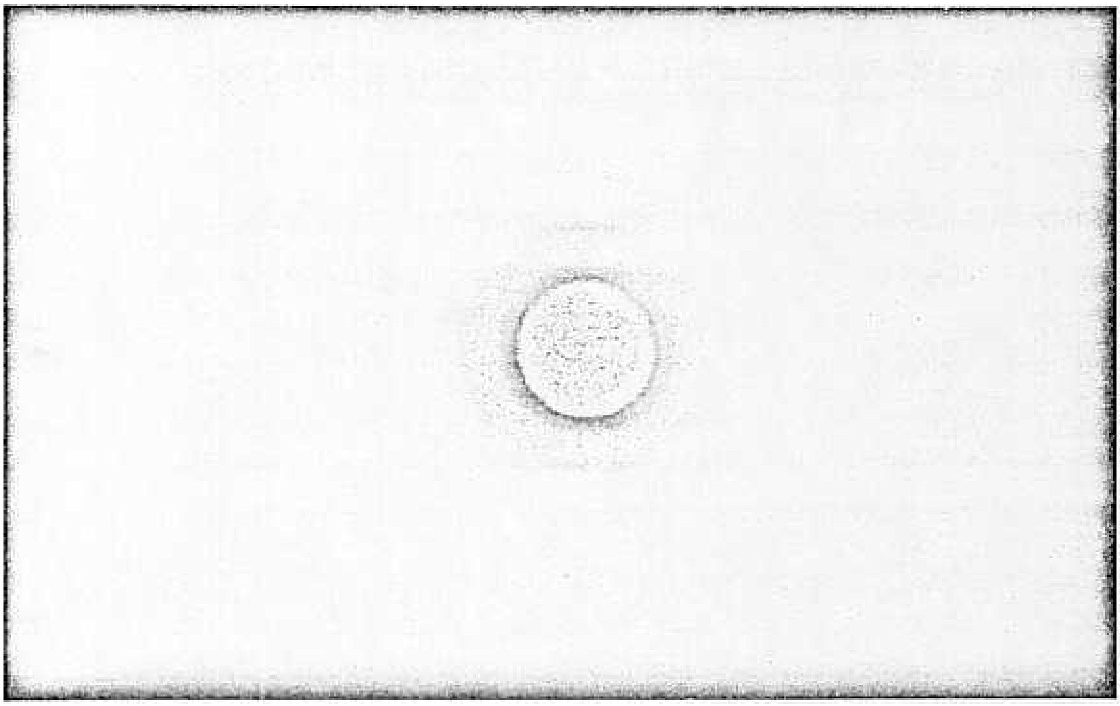


Fig. 13. (f) Fraunhofer pattern of this aperture.

pattern at each point of the array does change. Thus the scale of the array is governed by the mesh spacing, while the size of the individual patterns is governed by the size of the diffracting aperture (see Fig. 14).

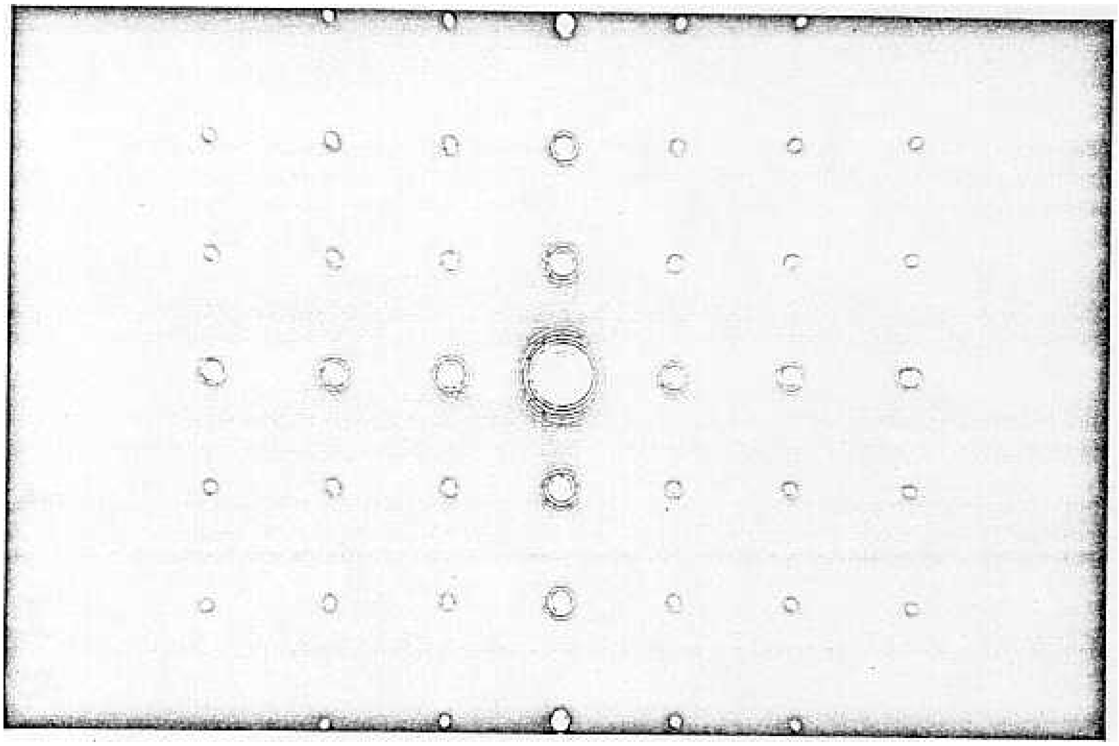
6. With the wire mesh and aperture in place, move L_2 slowly so that the Fresnel region can be observed Note how rapidly and dramatically the pattern changes in this region (see Fig. 15).

Photographs of a number of diffraction patterns not shown here may be found elsewhere (e.g., Refs. 2 and 3).

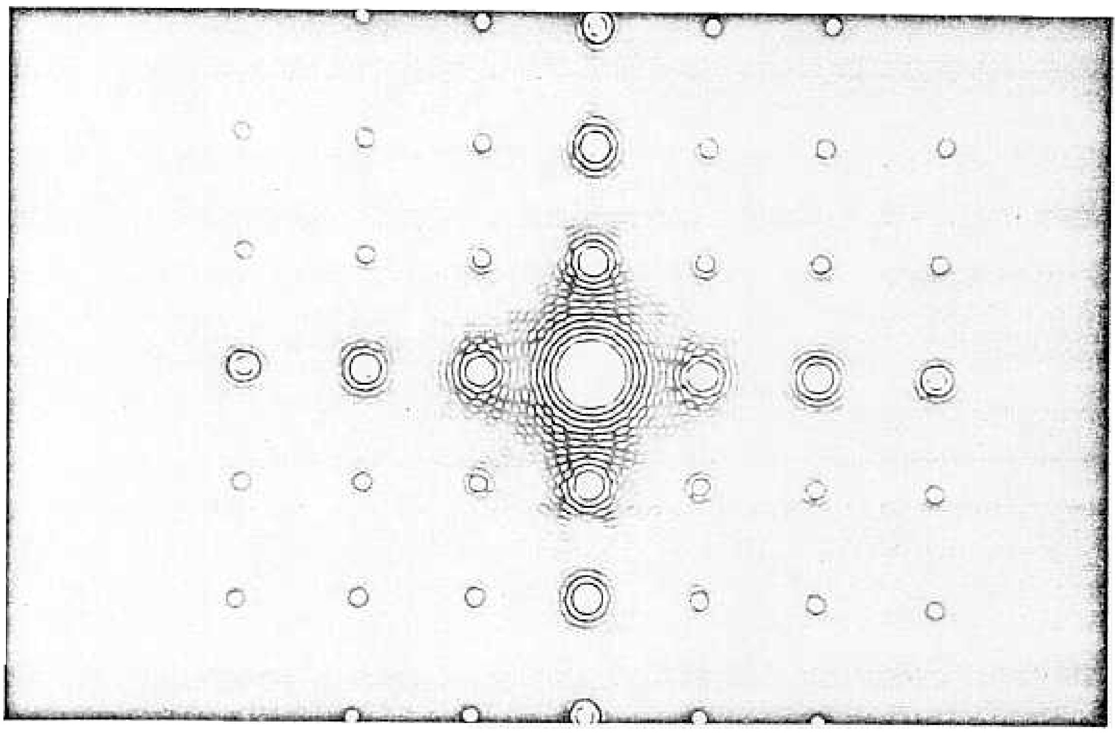
II. Spatial Filtering.

As given by (4), the complex amplitude in the Fraunhofer region of a diffracting aperture is the two-dimensional Fourier transform of the amplitude transmittance of that aperture. Thus this complex-amplitude distribution is proportional to the spatial-frequency spectrum of the aperture function. In an optical system, the form of an image can be changed by manipulating the spatial-frequency spectrum of the object. For example, consider the system shown in Fig. 7. If we place some semi-transparent object at the plane $z = 0$ and position L_2 such that an image of the object is cast onto the observation screen, we can significantly change the appearance of the image by placing various "spatial filters" in the Fraunhofer plane

14. Fraunhofer diffraction pattern of 10-wire/mm rectangular mesh when the limiting aperture is circular and has a diameter of

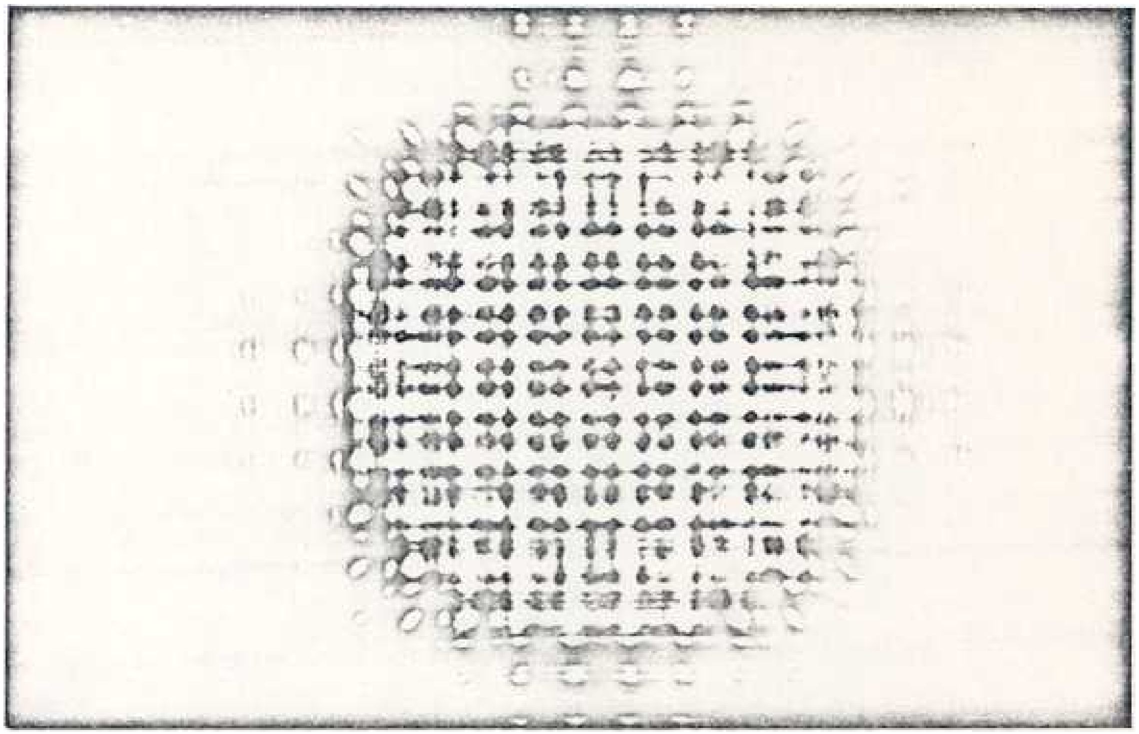


2.7 mm

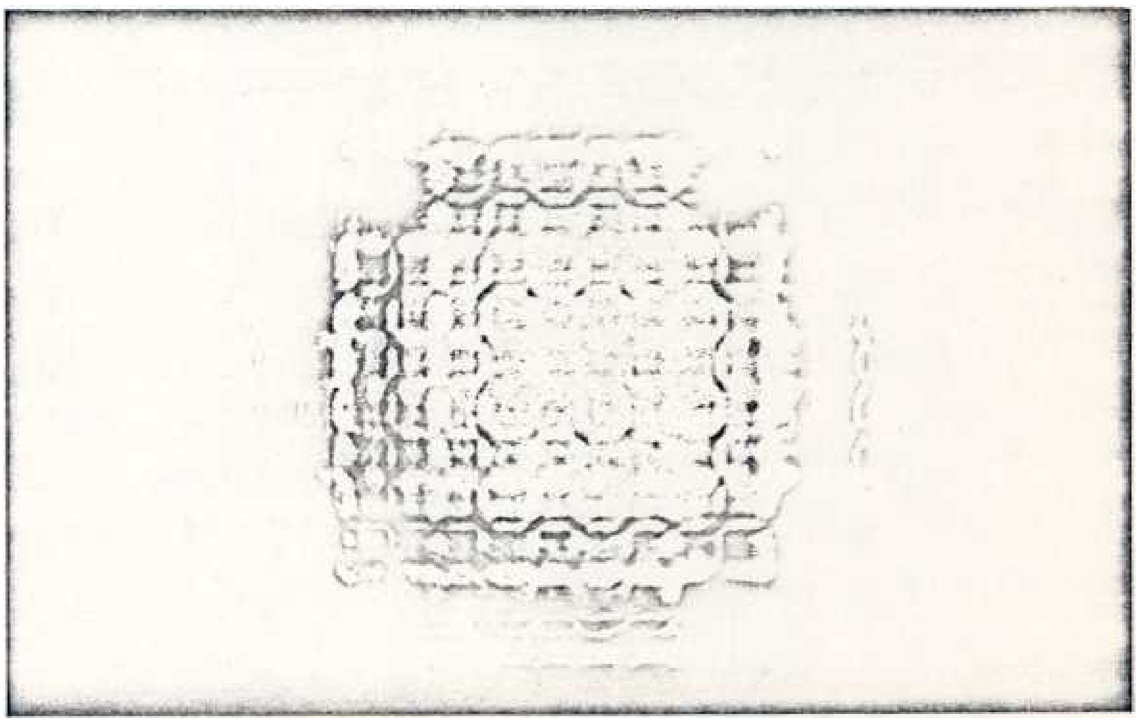


(b) 1.7 mm.

Fig. 15. Fresnel diffraction patterns:



(a) 10-wire/mm rectangular mesh with circular limiting aperture of 1.0-mm diameter



(b) coarse nonrectangular mesh but with same limiting aperture as in (a).

A. Background.

To see how this is done without getting bogged down in the mathematics, let us assume that our object is the mesh/aperture combination depicted in Fig. 16. As you observed in Part I of this demonstration, the Fraunhofer diffraction pattern of this combination consists of an array of bright spots (see Fig. 17). The transmittance function of the mesh, being periodic, can be decomposed into a linear combination of sinusoidal components with harmonically-related frequencies in both the x- and the y-directions. The various bright spots of the Fraunhofer pattern are related to the Fourier transforms of these sinusoidal components: the central spot corresponds to the zero-frequency component of the mesh transmittance; in any horizontal row, the spots immediately to the left and right of center correspond to the fundamental component of the transmittance function in the x-direction; the second pair of spots in this row correspond to the second harmonic component in the x-direction; etc. The same behavior is found for any vertical column of spots, except that these spots correspond to the various harmonic components in the y-direction.

If we now place a slit "filter" in the Fraunhofer plane to block out all the light except the single horizontal row of spots along the ξ-axis, there will be no variations of image irradiance in the vertical direction. All of the harmonic components associated with the x-direction are passed, and so the image has

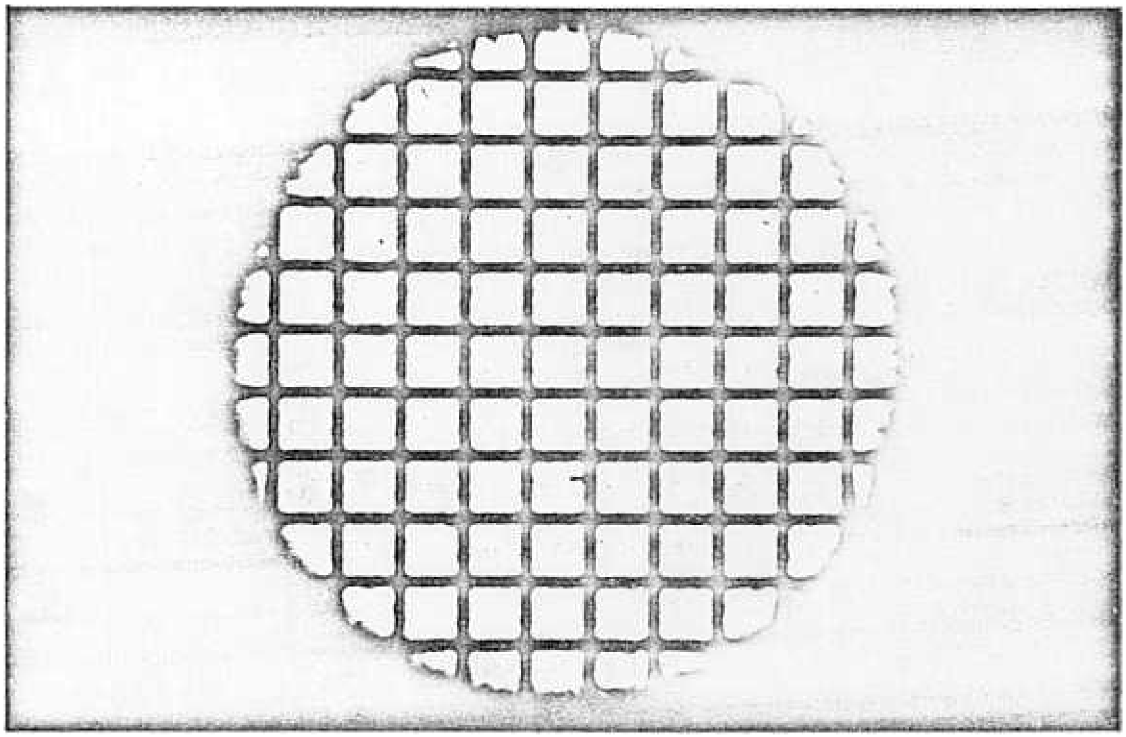


Fig. 16. Diffracting object for spatial-filtering demonstration consisting of 10-wire/mm mesh and limiting aperture of 1.0-mm diameter.

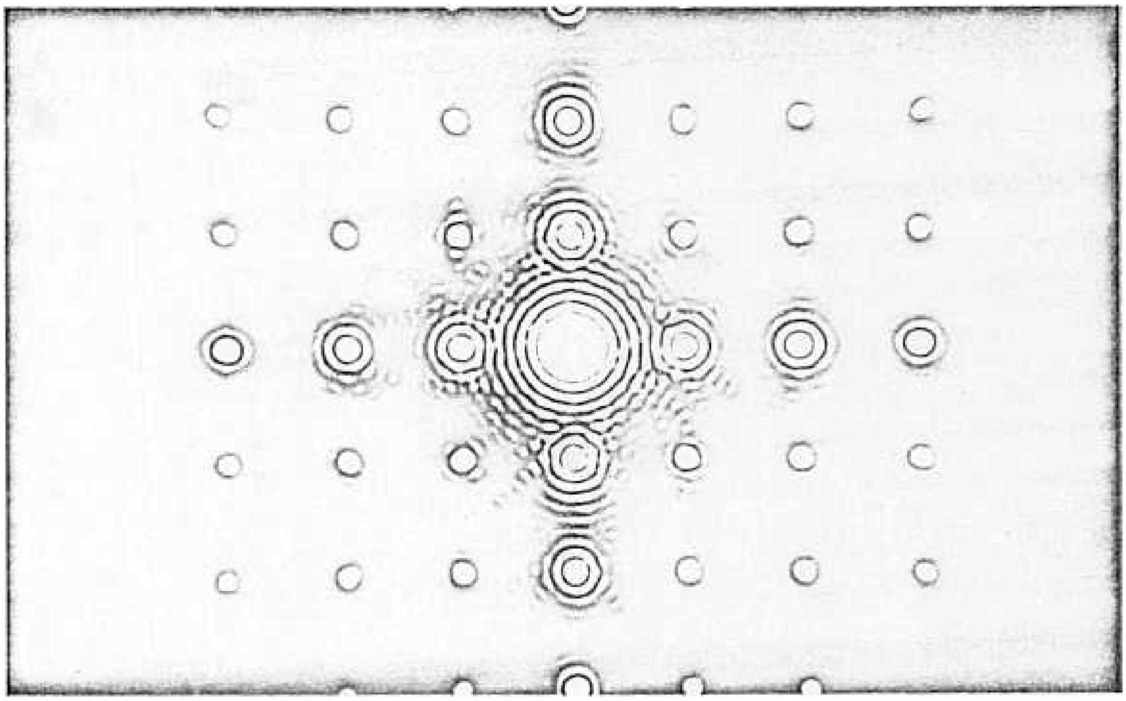


Fig. 17. Fraunhofer diffraction pattern of object shown in Fig. 16.

the behavior of the object in this direction. In the η-direction, however, we have eliminated all but the zero-frequency component, and thus the image irradiance must be constant in the y-direction.

To simplify the following development, we leave this slit filter in place so that we can work with quantities having variations only in the x-direction. The complex amplitude distribution in the image plane can then be approximated by the following Fourier series:

$$u_{im}(x,y) = [c_o + c_1\cos 2\pi\xi_o x + c_2\cos 2\pi(2\xi_o)x + c_3\cos 2\pi(3\xi_o)x + \ldots]p(x,y)$$

where p(x,y) is the aperture function and ξ_o is the fundamental frequency of the mesh in the x-direction. Here we have neglected the system magnification to avoid unnecessary complexity. The image irradiance is then the squared magnitude of $u_{im}(x,y)$

$$E_{im}(x,y) = |u_{im}(x,y)|^2$$

If we "filter out" any pair of spots located symmetrically about the origin, we effectively eliminate the corresponding term of (16) from the image.

Example 4: Let the frequency-plane filter block out all but the central spot. This eliminates all terms from (16) except the c_o term so that

$$u_{im}(x,y) = c_o p(x,y)$$

and

$$E_{im}(x,y) = c_o{}^2 p^2(x,y). \tag{19}$$

Thus the aperture is imaged, but not the mesh. (Actually, the aperture will not be sharply imaged, but this example is quite instructive nevertheless.)

B. Demonstration of spatial filtering.

The system used for the diffraction demonstrations should be used for this part as well (see Fig. 7), and the mesh/aperture combination of Fig. 16 should be used for the diffracting object Some sort of filter holder must be devised to block out the various spots in the Fraunhofer plane. This can be done by placing a thin piece of metal, containing a large hole, in the Fraunhofer plane and then clipping some 3" × 5" cards to it as shown in Fig. 18.

Procedures.

With the system specified above, perform the following:

1. Filter out all but the central horizontal row of spots and observe that the image variations in the y-direction do indeed disappear, while those in the x-direction remain (see Fig. 19).
2. Now repeat step 1, but in addition block out the central spot with a piece of fine wire. Using (16) and (17), predict how the image should look and compare

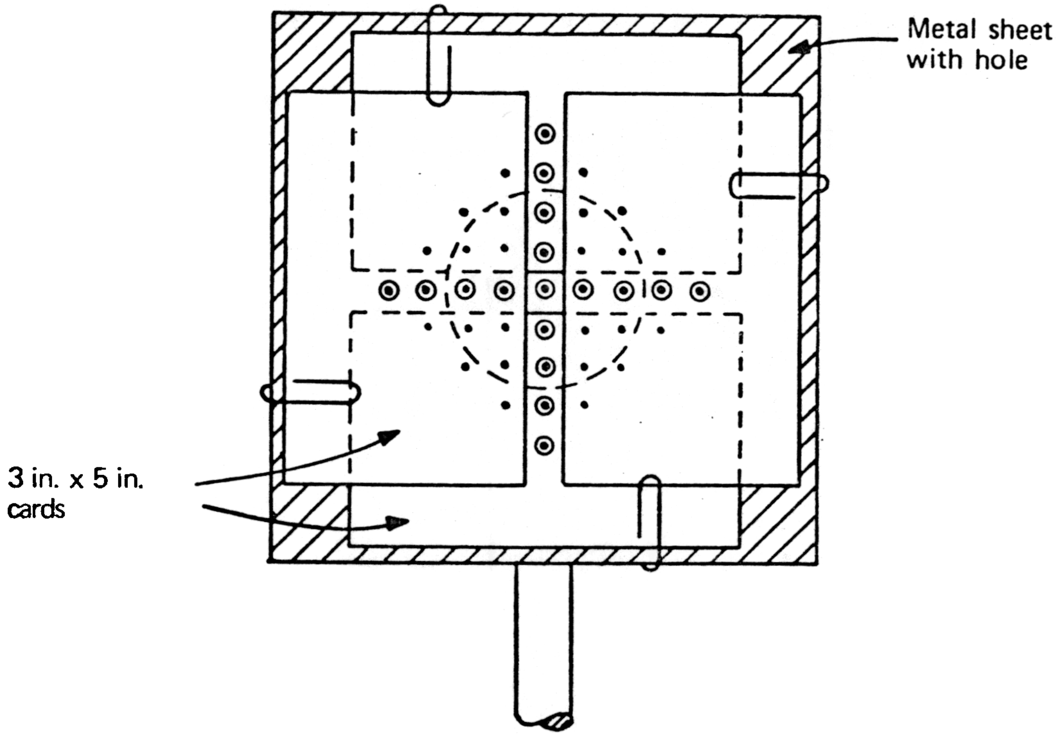


Fig. 18. Construction of simple spatial filter

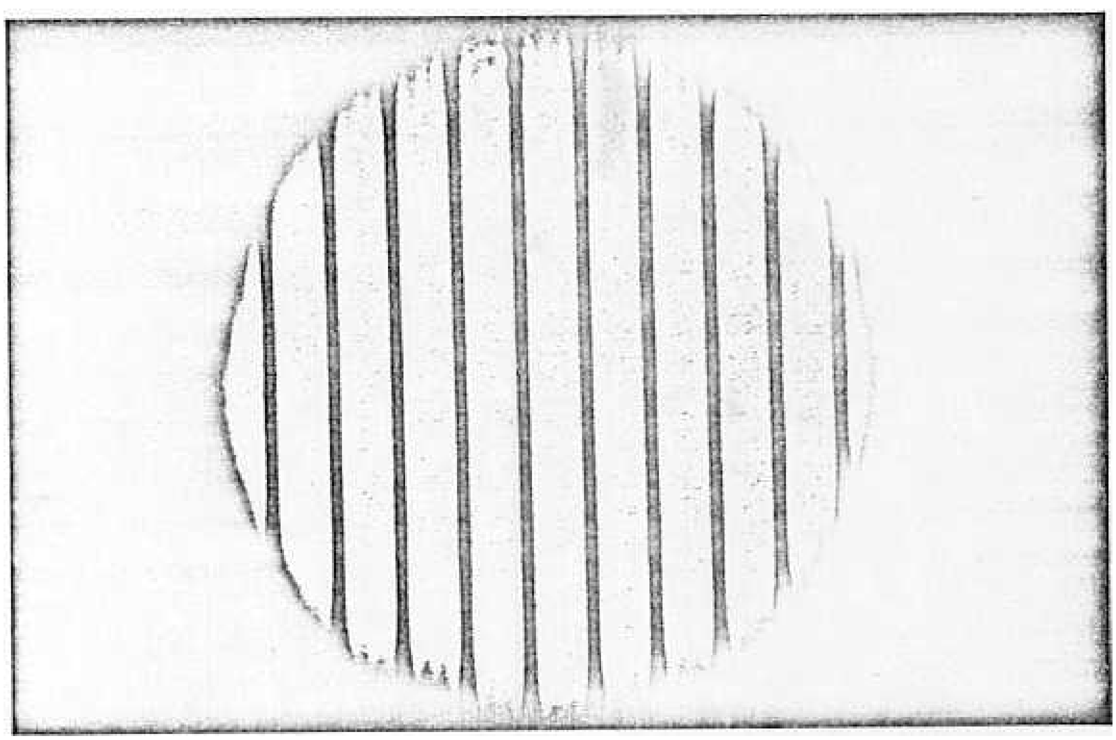


Fig. 19. Image of object with all vertical sinusoidal components filtered out.

and

$$E_{im}(x,y) = c_o^2 p^2(x,y). \tag{19}$$

Thus the aperture is imaged, but not the mesh. (Actually, the aperture will not be sharply imaged, but this example is quite instructive nevertheless.)

B. Demonstration of spatial filtering.

The system used for the diffraction demonstrations should be used for this part as well (see Fig. 7), and the mesh/aperture combination of Fig. 16 should be used for the diffracting object Some sort of filter holder must be devised to block out the various spots in the Fraunhofer plane. This can be done by placing a thin piece of metal, containing a large hole, in the Fraunhofer plane and then clipping some 3" × 5" cards to it as shown in Fig. 18.

Procedures.

With the system specified above, perform the following:

1. Filter out all but the central horizontal row of spots and observe that the image variations in the y-direction do indeed disappear, while those in the x-direction remain (see Fig. 19)
2. Now repeat step 1, but in addition block out the central spot with a piece of fine wire. Using (16) and (17), predict how the image should look and compare

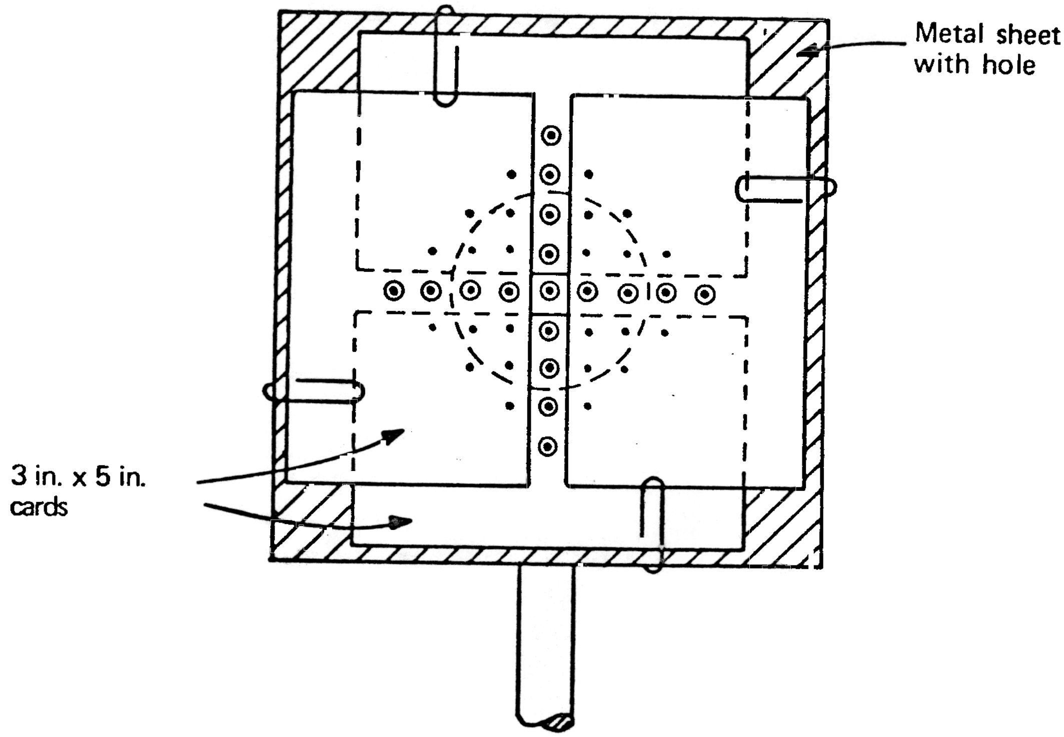


Fig. 18. Construction of simple spatial filter.

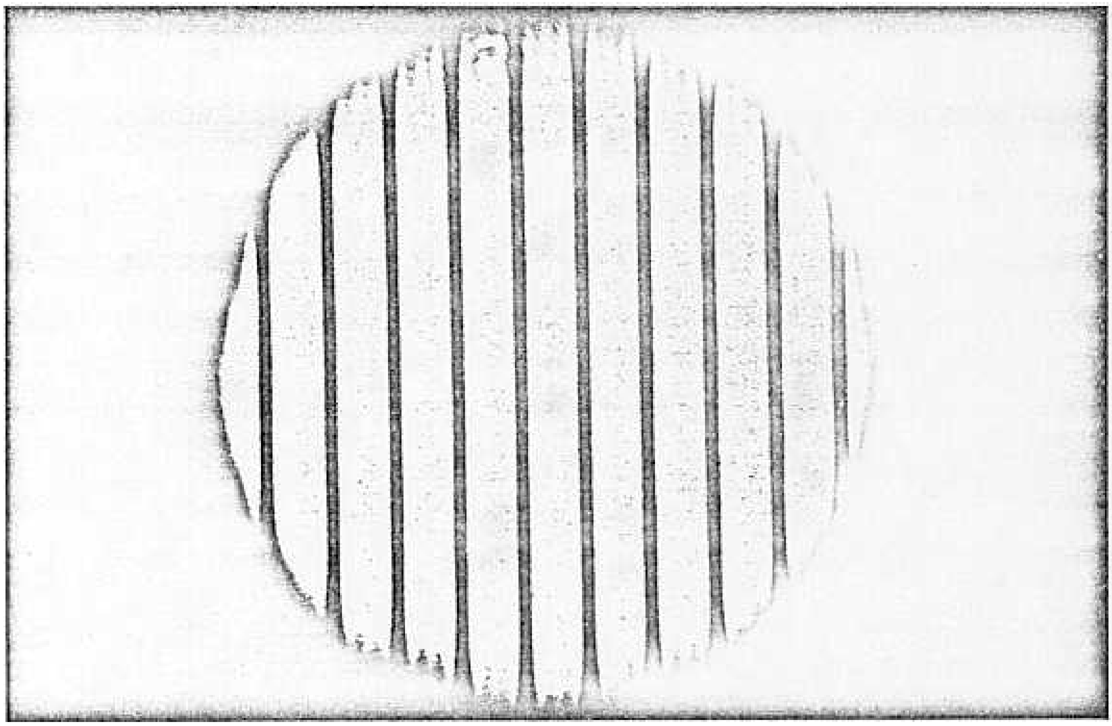


Fig. 19. Image of object with all vertical sinusoidal components filtered out.

this prediction with your observation (see Fig. 20) (Note that if the spacing of the mesh wires is several times larger than their diameter, then in (16) c_o is much greater than any of the other coefficients. The result is a "contrast reversal" effect--the bright areas become dark and the dark areas become light. Can you explain this?)

3. Now block all but the central spot and the first spot to either side of it. Using (16) and (17), predict how the image will look, and compare this prediction with your observation (see Fig. 21). Again note that if the spacing of the mesh wires is several times larger than their diameter, then $c_o >> c_1$ in (16).
4. Filter out all but the central spot and note how only the aperture appears in the image, but not the mesh, as contended in the preceding discussion (see Fig. 22).
5. Repeat step 3, but now block out the central spot with a piece of fine wire. The image should have the same form as in step 3 except that the frequency of the variations should have been doubled. Can you use (16) to explain this "frequency-doubling" effect?

In the above filtering demonstrations, only binary amplitude filters were used; such filters either pass all of the

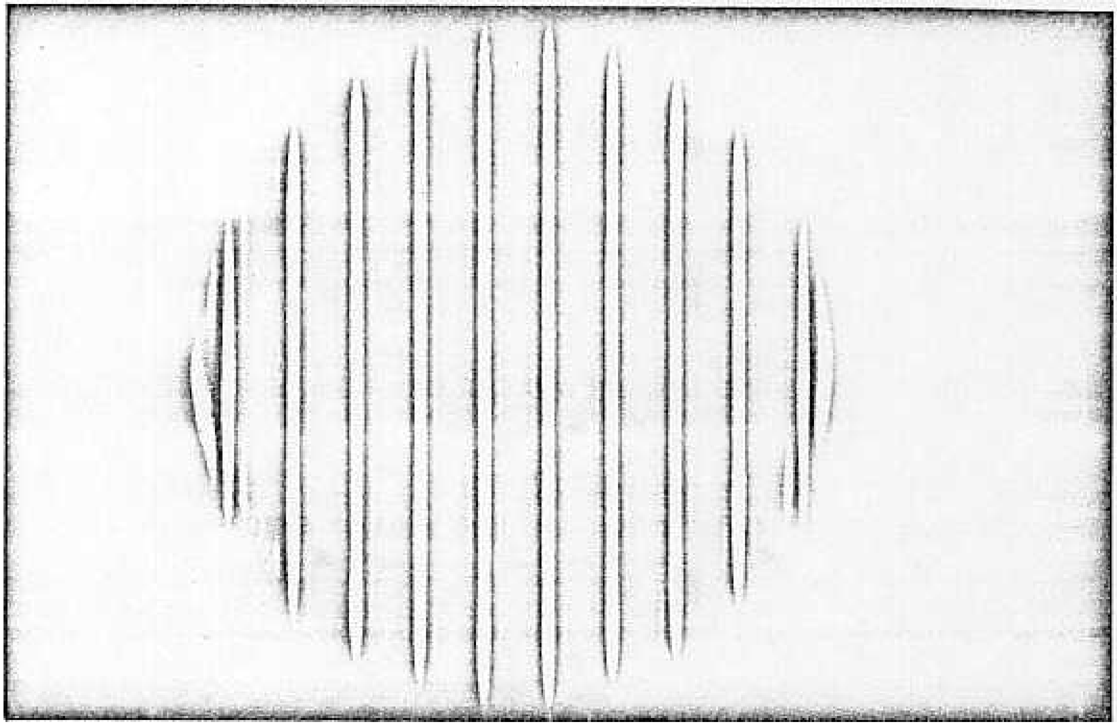


Fig. 20. Image of object with vertical sinusoidal components and zero-frequency component filtered out.

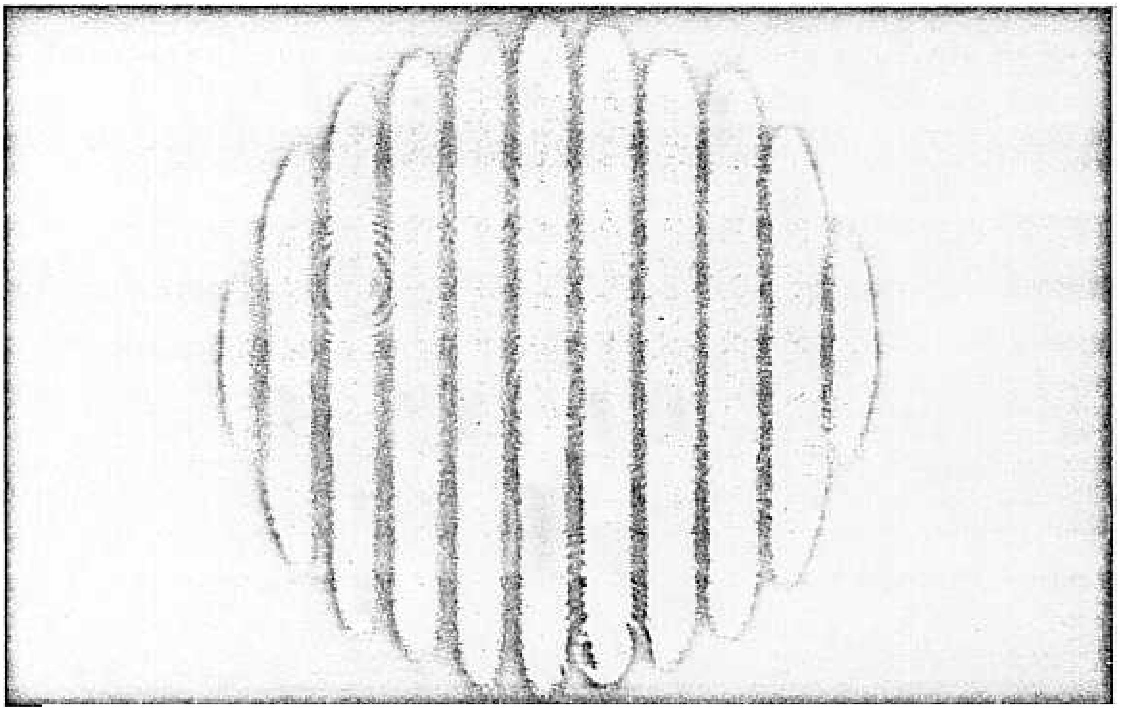


Fig. 21. Image of object with vertical components and all but the zero-frequency and fundamental horizontal components filtered out.

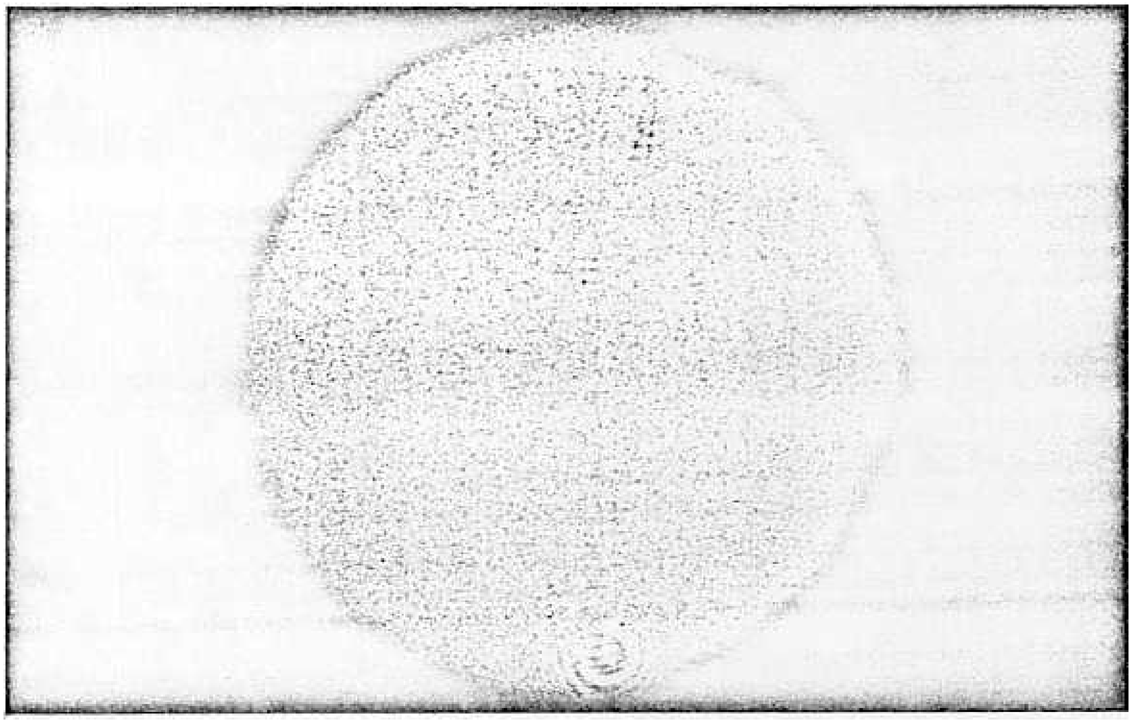


Fig. 22. Image of object when only the zero-frequency component is passed by the filter.

light incident at a point on the frequency plane, or block it completely. Phase filters alter the phase of this light, and complex filters, which are combinations of amplitude and phase filters, both attenuate and change the phase of the light. For our demonstration purposes amplitude filters were sufficient, but those who wish to find out more about phase filters and complex filters are referred to Ref. 1.

References.

1. J. W. Goodman, *Introduction to Fourier Optics*, McGraw-Hill, New York, 1968
2. G. B. Parrent and B. J. Thompson, *Physical Optics Notebook*, Society of Photo-optical Instrumentation Engineers, Redondo Beach, 1969.
3. G. Z. Dimitroff and J. G. Baker, *Telescopes and Accessories*, Blakiston, Philadelphia, 1945.