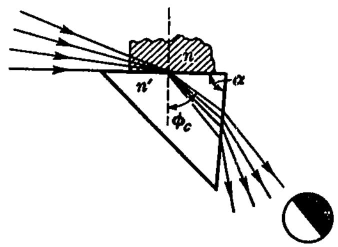

FIGURE 2D
Refraction by the prism in a Pulfrich refractometer.

two must be interchanged in Eq. (2a). The beam is so oriented that some of its rays just graze the surface (Fig. 2D) so that one observes in the transmitted light a sharp boundary between light and dark. Measurement of the angle at which this boundary occurs allows one to compute the value of ϕ_c and hence of n. There are important precautions that must be observed if the results are to be at all accurate.*

2.3 PLANE-PARALLEL PLATE

When a single ray traverses a glass plate with plane surfaces that are parallel to each other, it emerges parallel to its original direction but with a lateral displacement d which increases with the angle of incidence ϕ. Using the notation shown in Fig. 2E, we may apply the law of refraction and some simple trigonometry to find the displacement d. Starting with the right triangle ABE, we can write

$$d = l \sin (\phi - \phi') \tag{2b}$$

which, by the trigonometric relation for *the sine of the difference between two angles*, can be written

$$d = l(\sin \phi \cos \phi' - \sin \phi' \cos \phi) \tag{2c}$$

From the right triangle ABC we can write

$$l = \frac{t}{\cos \phi'}$$

which, substituted in Eq. (2c), gives

$$d = t\left(\frac{\sin \phi \cos \phi'}{\cos \phi'} - \frac{\sin \phi' \cos \phi}{\cos \phi'}\right) \tag{2d}$$

From Snell's law [Eq. (1m)] we obtain

$$\sin \phi' = \frac{n}{n'} \sin \phi$$

* For a valuable description of this and other methods of determining indices of refraction see A. C. Hardy and F. H. Perrin, "Principles of Optics," pp. 359–364, McGraw-Hill Book Company, New York, 1932.

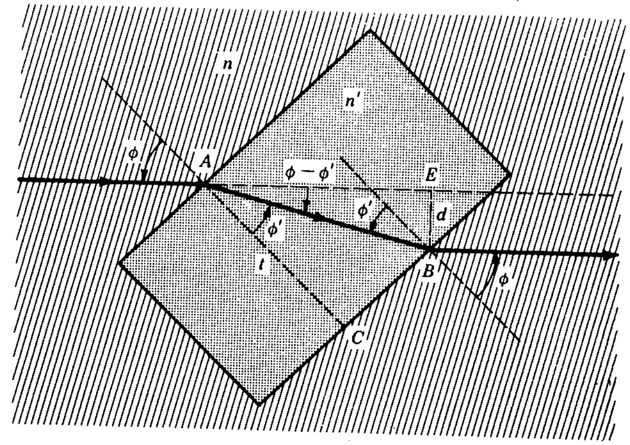

FIGURE 2E
Refraction by a plane-parallel plate.

which upon substitution in Eq. (2d), gives

$$d = t\left(\sin \phi - \frac{\cos \phi}{\cos \phi'}\frac{n}{n'} \sin \phi\right)$$

$$d = t \sin \phi \left(1 - \frac{n}{n'}\frac{\cos \phi}{\cos \phi'}\right) \tag{2e}$$

From 0° up to appreciably large angles, d is nearly proportional to ϕ, for as the ratio of the cosines becomes appreciably less than 1, causing the right-hand factor to increase, the sine factor drops below the angle itself in almost the same proportion.*

2.4 REFRACTION BY A PRISM

In a prism the two surfaces are inclined at some angle α so that the deviation produced by the first surface is not annulled by the second but is further increased. The chromatic dispersion (Sec. 1.10) is also increased, and this is usually the main function of a prism. First let us consider, however, the geometrical optics of the prism for light of a single color, i.e., for *monochromatic light* such as is obtained from a sodium arc.

* This principle is made use of in most of the home moving-picture film-editor devices in common use today. Instead of starting and stopping intermittently, as it does in the normal film projector, the film moves smoothly and continuously through the film-editor gate. A small eight-sided prism, immediately behind the film, produces a stationary image of each picture on the viewing screen of the editor. See Prob. 2.2 at the end of this chapter.

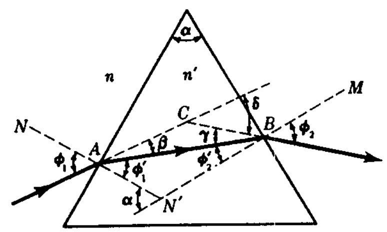

FIGURE 2F
The geometry associated with refraction by a prism.

The solid ray in Fig. 2F shows the path of a ray incident on the first surface at the angle ϕ_1.

Its refraction at the second surface, as well as at the first surface, obeys Snell's law, so that in terms of the angles shown

$$\frac{\sin \phi_1}{\sin \phi_1'} = \frac{n'}{n} = \frac{\sin \phi_2}{\sin \phi_2'} \qquad (2f)$$

The angle of deviation produced by the first surface is $\beta = \phi_1 - \phi_1'$, and that produced by the second surface is $\gamma = \phi_2 - \phi_2'$. The total angle of deviation δ between the incident and emergent rays is given by

$$\delta = \beta + \gamma \qquad (2g)$$

Since NN' and MN' are perpendicular to the two prism faces, α is also the angle at N'. From triangle ABN' and the exterior angle α, we obtain

$$\alpha = \phi_1' + \phi_2' \qquad (2h)$$

Combining the above equations, we obtain

$$\delta = \beta + \gamma = \phi_1 - \phi_1' + \phi_2 - \phi_2' = \phi_1 + \phi_2 - (\phi_1' + \phi_2')$$

or

$$\delta = \phi_1 + \phi_2 - \alpha \qquad (2i)$$

2.5 MINIMUM DEVIATION

When the total *angle of deviation* δ for any given prism is calculated by the use of the above equations, it is found to vary considerably with the angle of incidence. The angles thus calculated are in exact agreement with the experimental measurements. If during the time a ray of light is refracted by a prism the prism is rotated continuously in one direction about an axis (A in Fig. 2F) parallel to the refracting edge, the angle of deviation δ will be observed to decrease, reach a minimum, and then increase again, as shown in Fig. 2G.

The smallest deviation angle, called the *angle of minimum deviation* δ_m, occurs at that particular angle of incidence where the refracted ray inside the prism makes equal angles with the two prism faces (see Fig. 2H). In this special case

$$\phi_1 = \phi_2 \qquad \phi_1' = \phi_2' \qquad \beta = \gamma \qquad (2j)$$

To prove these angles equal, assume ϕ_1 does not equal ϕ_2 when minimum deviation occurs. By the principle of the reversibility of light rays (see Sec. 1.8),

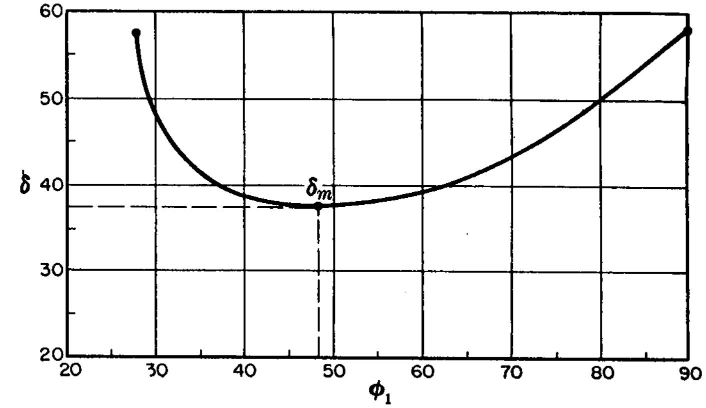

FIGURE 2G
A graph of the deviation produced by a 60° glass prism of index $n' = 1.50$. At minimum deviation $\delta_m = 37.2°$, $\phi_1 = 48.6°$, and $\phi_1' = 30.0°$.

there would be two different angles of incidence capable of giving minimum deviation. Since experimentally we find only one, there must be symmetry and the above equalities must hold.

In the triangle ABC in Fig. 2H the exterior angle δ_m equals the sum of the opposite interior angles $\beta + \gamma$. Similarly, for the triangle ABN', the exterior angle α equals the sum $\phi_1' + \phi_2'$. Consequently

$$\alpha = 2\phi_1' \qquad \delta_m = 2\beta \qquad \phi_1 = \phi_1' + \beta$$

Solving these three equations for ϕ_1' and ϕ_1 gives

$$\phi_1' = \tfrac{1}{2}\alpha \qquad \phi_1 = \tfrac{1}{2}(\alpha + \delta_m)$$

Since by Snell's law $n'/n = (\sin \phi_1)/(\sin \phi_1')$,

$$\frac{n'}{n} = \frac{\sin \frac{1}{2}(\alpha + \delta_m)}{\sin \frac{1}{2}\alpha} \qquad (2k)$$

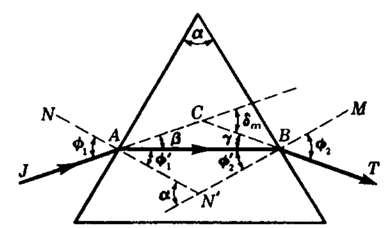

FIGURE 2H
The geometry of a light ray traversing a prism at minimum deviation.

The most accurate measurements of refractive index are made by placing the sample in the form of a prism on the table of a spectrometer and measuring the angles δ_m and α, the former for each color desired. When prisms are used in spectroscopes and spectrographs, they are always set as nearly as possible at minimum deviation because otherwise any slight divergence or convergence of the incident light would cause astigmatism in the image.

2.6 THIN PRISMS

The equations for the prism become much simpler when the refracting angle α becomes small enough to ensure that its sine and the sine of the angle of deviation δ may be set equal to the angles themselves. Even at an angle of 0.1 rad, or 5.7°, the difference between the angle and its sine is less than 0.2 percent. For prisms having a refracting angle of only a few degrees, we can therefore simplify Eq. (2k) by writing

$$\frac{n'}{n} = \frac{\sin \frac{1}{2}(\delta_m + \alpha)}{\sin \frac{1}{2}\alpha} = \frac{\delta_m + \alpha}{\alpha}$$

● and

$$\delta = (n' - 1)\alpha \qquad (2l)$$

Thin prism in air

The subscript on δ has been dropped because such prisms are always used at or near minimum deviation, and n has been dropped because it will be assumed that the surrounding medium is air, $n = 1$.

It is customary to measure the *power* of a prism by the deflection of the ray in centimeters at a distance of 1 m, in which case the unit of power is called the *prism diopter* (D). A prism having a power of 1 prism diopter therefore displaces the ray on a screen 1 m away by 1 cm. In Fig. 2I(*a*) the deflection on the screen is x cm and is numerically equal to the power of the prism. For small values of δ it will be seen that the power in prism diopters is essentially the angle of deviation δ measured in units of 0.01 rad, or 0.573°.

For the dense flint glass of Table 1A, $n'_D = 1.67050$, and Eq. (2l) shows that the refracting angle of a 1-D prism should be

$$\alpha = \frac{0.57300}{0.67050} = 0.85459°$$

2.7 COMBINATIONS OF THIN PRISMS

In measuring binocular accommodation, ophthalmologists make use of a combination of two thin prisms of equal power which can be rotated in opposite directions in their own plane [Fig. 2I(*b*)]. Such a device, known as the *Risley* or *Herschel prism*, is equivalent to a single prism of variable power. When the prisms are parallel, the power is twice that of either one; when they are opposed, the power is zero. To find how the power and direction of deviation depend on the angle between the

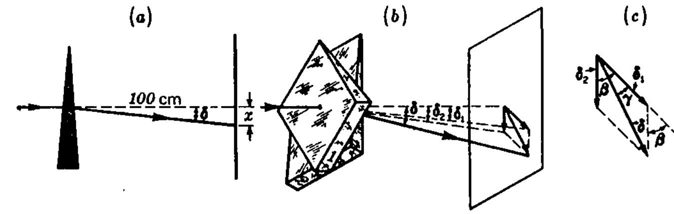

FIGURE 2I
Thin prisms: (*a*) the displacement x in centimeters at a distance of 1 m gives the power of the prism in diopters; (*b*) Risley prism of variable power; (*c*) vector addition of prism deviations.

components, we use the fact that the deviations add vectorially. In Fig. 2I(*c*) it will be seen that the resultant deviation δ will in general be, from the law of cosines,

$$\delta = \sqrt{\delta_1^2 + \delta_2^2 + 2\delta_1\delta_2 \cos \beta} \qquad (2m)$$

where β is the angle between the two prisms. To find the angle γ between the resultant deviation and that due to prism 1 alone (or, we may say, between the "equivalent" prism and prism 1) we have the relation

$$\tan \gamma = \frac{\delta_2 \sin \beta}{\delta_1 + \delta_2 \cos \beta} \qquad (2n)$$

Since almost always $\delta_1 = \delta_2$, we may call the deviation by either component δ_i, and the equations simplify to

$$\delta = \sqrt{2\delta_i^2(1 + \cos \beta)} = \sqrt{4\delta_i^2 \cos^2 \frac{\beta}{2}} = 2\delta_i \cos \frac{\beta}{2} \qquad (2o)$$

and

$$\tan \gamma = \frac{\sin \beta}{1 + \cos \beta} = \tan \frac{\beta}{2}$$

so that

$$\gamma = \frac{\beta}{2} \qquad (2p)$$

2.8 GRAPHICAL METHOD OF RAY TRACING

It is often desirable in the process of designing optical instruments to be able to trace rays of light through the system quickly. For prism instruments the principles presented below are extremely useful. Consider first a 60° prism of index $n' = 1.50$ surrounded by air of index $n = 1.00$. After the prism has been drawn to scale, as in Fig. 2J, and the angle of incidence ϕ_1 has been selected, the construction begins as in Fig. 1G.

Line OR is drawn parallel to JA, and, with an origin at O, the two circular arcs are drawn with radii proportional to n and n'. Line RP is drawn parallel to NN',

23 DISPERSION

JENKINS & WHITE
FUNDAMENTALS OF OPTICS
1976, McGRAW-HILL

The subject of dispersion concerns the speed of light in material substances and its variation with wavelength. Since the speed is c/n, any change in refractive index n entails a corresponding change of speed. We have seen in Sec. 1.4 that the dispersion of color which occurs upon refraction at a boundary between two different substances is direct evidence of the dependence of the n's on wavelength. In fact, measurements of the deviations of several spectral lines by a prism furnish the most accurate means of determining the refractive index, and hence the speed, as a function of wavelength.

23.1 DISPERSION OF A PRISM

When a ray traverses a prism, as shown in Fig. 23A, we can measure with a spectrometer the angles of emergence θ of the various wavelengths. The rate of change, $d\theta/d\lambda$, is called the *angular dispersion* of the prism. It can be conveniently represented as the product of two factors, by writing

$$\frac{d\theta}{d\lambda} = \frac{d\theta}{dn}\frac{dn}{d\lambda} \qquad (23a)$$

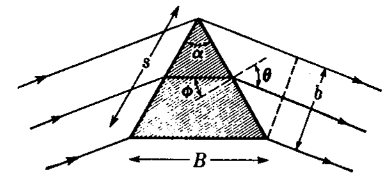

FIGURE 23A
Refraction by a prism at minimum deviation.

The first factor can be evaluated by geometrical considerations alone, while the second is a characteristic property of the prism material, usually referred to simply as its *dispersion*. Before considering the latter quantity, let us evaluate the geometrical factor $d\theta/dn$ for a prism, in the special case of minimum deviation.

For a given angle of incidence on the second face of the prism, we differentiate Snell's law of refraction $n = \sin\theta/\sin\phi$, regarding $\sin\phi$ as a constant, and obtain

$$\frac{d\theta}{dn} = \frac{\sin\phi}{\cos\theta}$$

This is not, however, the value to be used in Eq. (23a), which requires the rate of change of θ for a fixed direction of the rays incident on the *first* face. Because of the symmetry in the case of minimum deviation, it is obvious that equal deviations occur at the two faces, so that the total rate of change will be just twice the above value. We then have

$$\frac{d\theta}{dn} = \frac{2\sin\phi}{\cos\theta} = \frac{2\sin(\alpha/2)}{\cos\theta}$$

where α is the refracting angle of the prism. The result becomes still simpler when expressed in terms of lengths rather than angles. Designating by s, B, and b the lengths shown in Fig. 23A, we write

$$\frac{d\theta}{dn} = \frac{2s\sin(\alpha/2)}{s\cos\theta} = \frac{B}{b} \qquad (23b)$$

Hence the required geometrical factor is just the ratio of the base of the prism to the linear aperture of the emergent beam, a quantity not far different from unity. The angular dispersion becomes

● $$\frac{d\theta}{d\lambda} = \frac{B}{b}\frac{dn}{d\lambda} \qquad (23c)$$

In connection with this equation, it is to be noted that the equation for the chromatic resolving power [Eq. (15j)] follows very simply from it upon the substitution of λ/b for $d\theta$.

23.2 NORMAL DISPERSION

In considering the second factor in Eq. (23a), let us start by reviewing some of the known facts about the variation of n with λ. Measurements for some typical kinds of glass give the results shown in Tables 23A and 23B. If any set of values of n is plotted against wavelength, a curve like one of those in Fig. 23B is obtained. The curves found for prisms of different optical materials will differ in detail but will all have the same general shape. These curves are representative of *normal dispersion*, for which the following important facts are to be noted:

1 *The index of refraction increases as the wavelength decreases.*
2 *The rate of increase becomes greater at shorter wavelengths.*
3 *For different substances the curve at a given wavelength is usually steeper the larger the index of refraction.*
4 *The curve for one substance cannot in general be obtained from that for another substance by a mere change in the scale of the ordinates.*

The first of these facts agrees with the common observation that in refraction by a transparent substance the violet is more deviated than the red. The second fact can also be expressed by saying that the dispersion increases with decreasing wavelength. This follows because the dispersion $dn/d\lambda$ is the slope of the curve (its negative

Table 23A REFRACTIVE INDEX FOR SEVERAL TRANSPARENT SOLIDS

Substance	Violet 4100	Blue 4700	Green 5500	Yellow 5800	Orange 6100	Red 6600
	Color wavelength λ, Å					
Crown glass	1.5380	1.5310	1.5260	1.5225	1.5216	1.5200
Light flint	1.6040	1.5960	1.5910	1.5875	1.5867	1.5850
Dense flint	1.6980	1.6836	1.6738	1.6670	1.6650	1.6620
Quartz	1.5570	1.5510	1.5468	1.5438	1.5432	1.5420
Diamond	2.4580	2.4439	2.4260	2.4172	2.4150	2.4100
Ice	1.3170	1.3136	1.3110	1.3087	1.3080	1.3060
Strontium titanate ($SrTiO_3$)	2.6310	2.5106	2.4360	2.4170	2.3977	2.3740
Rutile (TiO_2), E ray	3.3408	3.1031	2.9529	2.9180	2.8894	2.8535

Table 23B REFRACTIVE INDICES AND DISPERSIONS FOR SEVERAL COMMON TYPES OF OPTICAL GLASS
Unit of dispersion 1/Å × 10^{-5}

Wavelength λ, Å	Telescope crown n	Telescope crown $-\frac{dn}{d\lambda}$	Borosilicate crown n	Borosilicate crown $-\frac{dn}{d\lambda}$	Barium flint n	Barium flint $-\frac{dn}{d\lambda}$	Vitreous quartz n	Vitreous quartz $-\frac{dn}{d\lambda}$
C 6563	1.52441	0.35	1.50883	0.31	1.58848	0.38	1.45640	0.27
6439	1.52490	0.36	1.50917	0.32	1.58896	0.39	1.45674	0.28
D 5890	1.52704	0.43	1.51124	0.41	1.59144	0.50	1.45845	0.35
5338	1.52989	0.58	1.51386	0.55	1.59463	0.68	1.46067	0.45
5086	1.53146	0.66	1.51534	0.63	1.59644	0.78	1.46191	0.52
F 4861	1.53303	0.78	1.51690	0.72	1.59825	0.89	1.46318	0.60
G′ 4340	1.53790	1.12	1.52136	1.00	1.60367	1.23	1.46690	0.84
H 3988	1.54245	1.39	1.52546	1.26	1.60870	1.72	1.47030	1.12

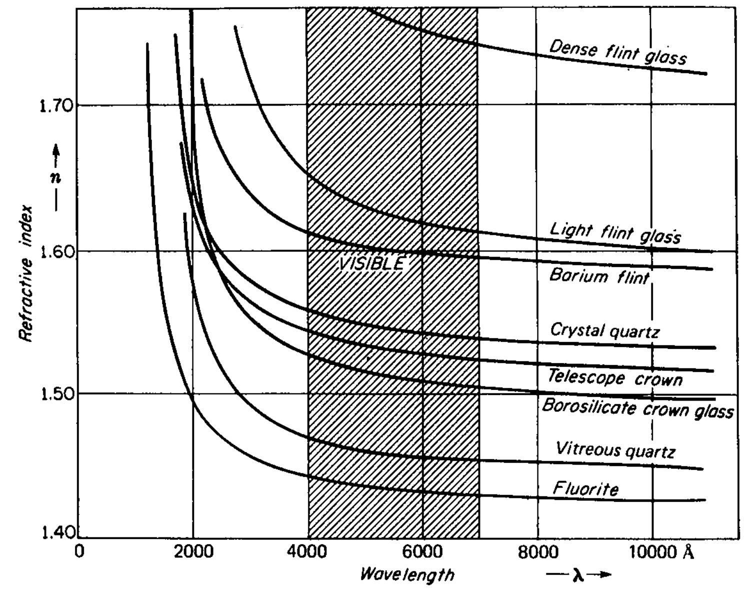

FIGURE 23B
Dispersion curves for several different materials commonly used for lenses and prisms.

sign is usually disregarded), which increases regularly toward smaller λ. An important consequence of this behavior of the dispersion is that in the spectrum formed by a prism the violet end of the spectrum is spread out on a much larger scale than the red end. The spectrum is therefore far from being a normal spectrum (Sec. 17.6). This will be clear from Fig. 23C, in which the spectrum of helium is shown diagrammatically as given by flint- and crown-glass prisms and by a grating used under the proper conditions to give a normal spectrum. In the prism spectra the wavelength scale is compressed toward the red end, as can be seen by comparison with the uniform scale of the normal spectrum.

The third fact stated above requires that for a substance of higher index of refraction, the dispersion $dn/d\lambda$ shall also be greater. Thus, comparing (*a*) and (*b*) in Fig. 23C, the flint glass has the higher index of refraction and gives a longer spectrum because of its greater dispersion. To compare the *relative* spacing of the lines in (*b*) with those in (*a*), the spectrum from crown glass has been enlarged, in (*c*), to have the same overall length between the two lines λ3888 and λ6678. When this is done, it is seen that there is not complete agreement with the lines of (*a*). In fact, the spectra from prisms of different substances will never agree exactly in the relative spacing of their spectrum lines. This is a consequence of the fourth of the above facts,

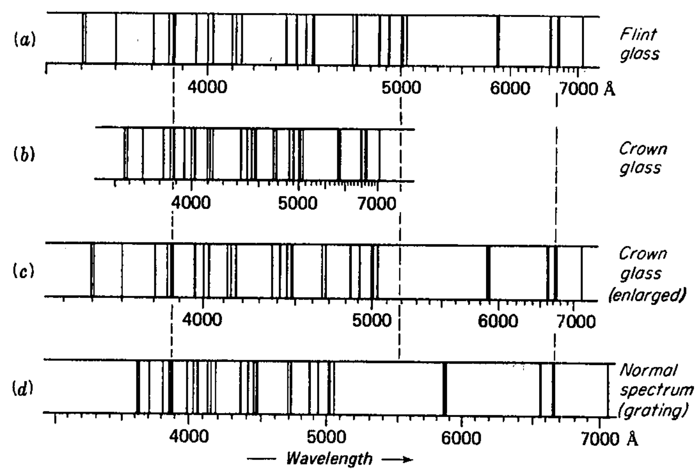

FIGURE 23C
Comparison of the helium spectrum produced by flint-glass and crown-glass prism spectrographs with a normal spectrum.

according to which the shape of the dispersion curve is different for every substance. The curve for flint glass in Fig. 23B has a greater slope at the violet end, relative to that in the red, than does the curve for crown glass. Consequently, the dispersion of different substances is said to be *irrational*, since there is no simple relation between the different curves.

All transparent substances which are not colored show normal dispersion in the visible region. The magnitude of the index of refraction may be quite different in various substances, but its change with wavelength always shows the characteristics described above. In general, the greater the density of the substance the higher its index of refraction and its dispersion. For example, flint glass has a density around 2.8, considerably higher than 2.4 for ordinary crown glass. Water has a smaller n and $dn/d\lambda$, while in a very light substance like air n is practically unity and $dn/d\lambda$ very nearly zero. For air $n = 1.000276$ for red light (Fraunhofer's C line), rising to only 1.000279 for blue light (F line). This rule relating density to index of refraction is only a qualitative one, and many exceptions are known. For instance, ether has a higher index than water (1.36 as compared with 1.33), yet it is less dense, as is shown by the fact that ether floats on the surface of water. Similarly, the correlation of high dispersion with high index is only rough, and there are exceptions to the third rule listed above. Diamond has a density of 3.52 and one of the highest known indices of refraction, varying from 2.4100 for the C line to 2.4354 for the F line. The difference in these values, which is a measure of the dispersion, is only 0.0254, whereas a dense flint glass may give as much as 0.05 for the same quantity.

23.3 CAUCHY'S EQUATION

The first successful attempt to represent the curve of normal dispersion by an equation was made by Cauchy in 1836. His equation may be written

$$n = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4}$$

where A, B, and C are constants which are characteristic of any one substance. This equation represents the curves in the visible region, such as those shown in Fig. 23B, with considerable accuracy. To find the values of the three constants, it is necessary to know values of n for three different λ's. Then three equations may be set up which, when solved as simultaneous equations, give A, B, and C. For some purposes it is sufficiently accurate to include only the first two terms and the two constants can be found from values of n at only two λ's. The two-constant Cauchy equation is, then,

● $$n = A + \frac{B}{\lambda^2} \qquad (23d)$$

from which the dispersion becomes, by differentiation

● $$\frac{dn}{d\lambda} = -\frac{2B}{\lambda^3} \qquad (23e)$$

This shows that the dispersion varies approximately as the inverse cube of the wavelength. At 4000 Å it will be about 8 times as large as at 8000 Å. The minus sign corresponds to the usual negative slope of the dispersion curve.

The theoretical reasoning on which Cauchy based his equation was later shown to be false, so that it is to be considered essentially as an empirical equation. Nevertheless it holds very satisfactorily for cases of normal dispersion and is a useful equation from a practical standpoint. We shall show later that it is a special case of a more complete equation which does have a sound theoretical foundation.

23.4 ANOMALOUS DISPERSION

If measurements of the index of refraction of a transparent substance like quartz are extended into the infrared region of the spectrum, the dispersion curve begins to show marked deviations from the Cauchy equation. The deviation is always of the type illustrated in Fig. 23D, where, starting at the point R, the index of refraction is seen to fall off more rapidly than required by a Cauchy equation that represents the values of n for visible light (between P and Q) quite accurately. This equation predicts a very gradual decrease of n for large values of λ (broken curve), the index approaching the limiting value A as λ approaches infinity [Eq. (23d)]. In contrast to this, the measured value of n first decreases more and more rapidly as it approaches a region in the infrared where light ceases to be transmitted at all. This is an absorption band (Sec. 22.3), i.e., a region of selective absorption, the position of which is characteristic of the material. Within the absorption band, n cannot usually be meas-