QUANTUM QUALIFYING EXAM
AUGUST 2007

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PROBLEM 1

The wave function of a particle of mass m in free space is approximated by $\phi(\vec{r}) = Ne^{i\vec{k}\cdot\vec{r}}$ where $\vec{k}$ is a constant and N is a normalization constant.

[a] (4 pts) What would be the result of a measurement of the momentum of the particle? Explain your answer.

[b] (4 pts) What would be the result of a measurement of the energy of the particle? Explain your answer.

[c] (2 pts) What would be the result of a measurement of the position of the particle? Explain your answer.

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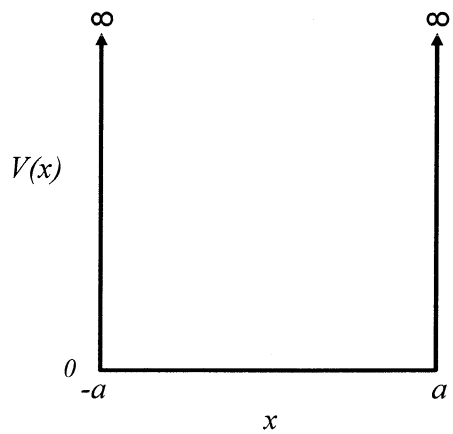
PROBLEM 2



Consider the one-dimensional infinite-well potential shown above.

[a] (4pts) Derive expressions for the energy eigenfunctions and energy eigenvalues for a particle in the one-dimensional infinite well shown above. Show your work.

[b] (4pts) Now suppose a perturbation of the form

$$\Delta V(x) = V_o a\, \delta(x)$$

is added with $V_o << \frac{\hbar^2 \pi^2}{ma^2}$. Here $\delta(x)$ is the Dirac-delta function. According to first order perturbation theory, what are the eigenenergies of each state?

[c] (2pts) According to first order perturbation theory, what is the wave function of the ground state? Write your answer in terms of a, V_o, fundamental constants, and the unperturbed wave functions $\phi_n(x)$. You do not have to normalize the wave function.

P2

a) $\psi_n(x) = \frac{1}{\sqrt{a}} \cos\left(\frac{n\pi x}{2L}\right)$ n odd

$\psi_n(x) = \frac{1}{\sqrt{a}} \sin\left(\frac{n\pi x}{2L}\right)$ n odd

$$E_n^{(1)} = \langle n^{(0)} | H' | n^{(0)} \rangle$$

$$= \int_{-a}^{+a} \psi_n^*(x)\, V_0 a \delta(x) \psi_n(x)$$

$$= \frac{V_0 a}{a} \int_{-a}^{+a} \cos\left(\frac{n\pi x}{2L}\right) \delta(x) \cos\left(\frac{n\pi x}{2L}\right) dx$$

$$= V_0 \cos^2 0 = V_0$$

$$E_n = E_n^{(0)} + E_n^{(1)}$$

$$= \frac{n^2\pi^2\hbar^2}{8ma^2} + V_0$$

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Have to verify the whole thing

PROBLEM 3
A particle of mass m has a potential energy represented by two one-dimensional harmonic oscillator potentials centered at $\pm a$:

$$V(x) = \frac{1}{2}K(x-a)^2 + \frac{1}{2}K(x+a)^2$$

[a] (3 pts) What are the eigenvalues of the particle given this potential V? You may derive this result from first principles or deduce the result from the well known eigenvalues of a particle moving in a single harmonic-oscillator potential.

[b] (3 pts) The normalized ground-state eignefunction of the particle is given by

$$\phi(x) = \frac{1}{\pi^{1/4}\Delta^{1/2}} \exp\left(-\frac{x^2}{2\Delta^2}\right)$$

Use Schrodinger's equation to determine the constant Δ in terms of K, m, and fundamental constants.

[c] (4 pt) The potential well at $x = -a$ suddenly disappears, leaving the particle in a new potential

$$U(x) = \frac{1}{2}K(x-a)^2$$

Suppose that *before the sudden change,* the particle was in the ground state of the double-well potential $V(x)$. Derive an expression for the probability that after the sudden change the particle will be in the ground state of the single well potential $U(x)$. Express your answer in terms of a and Δ.

ψ

ψ'

$r = a_0$

$\langle r \rangle = \frac{3}{2} a_0$

P3.

$$V(x) = \frac{1}{2} K (x-a)^2 + \frac{1}{2} K (x+a)^2$$

a) The eigenstate of the particle can be written as $|n\rangle$

$$H = \frac{\hat{p}^2}{2m} + V(x) = \frac{\hat{p}^2}{2m} + \frac{1}{2} K \left[(x-a)^2 + (x+a)^2 \right]$$

$$= \frac{\hat{p}^2}{2m} + \frac{1}{2} K x^2 + K a^2$$

$$\begin{cases} a^+ = \sqrt{\frac{m\omega}{2\hbar}} \hat{x} - i \frac{\hat{p}}{\sqrt{2m\omega\hbar}} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i\hat{p}}{m\omega} \right) \\ a = \sqrt{\frac{m\omega}{2\hbar}} \hat{x} + i \frac{\hat{p}}{\sqrt{2m\omega\hbar}} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i\hat{p}}{m\omega} \right) \\ \hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (a + a^+) \qquad \hat{p} = -i \sqrt{\frac{\hbar m\omega}{2}} (a - a^+) \end{cases}$$

$$H = -\frac{\hbar m\omega}{2} \frac{1}{2m} (a-a^+)^2 + \frac{1}{2} K \frac{\hbar}{2m\omega} (a+a^+)^2 + K a^2$$

$$= -\frac{\hbar\omega}{4} \left[(a)^2 + (a^+)^2 - a a^+ - a^+ a \right] + \frac{\hbar\omega}{4} \left[(a)^2 + (a^+)^2 + a a^+ + a^+ a \right] + K a^2$$

$$= \frac{\hbar\omega}{2} (a a^+ + a^+ a) + K \alpha^2$$

$$H|n\rangle = \left\{ \frac{\hbar\omega}{2} (a a^+ + a^+ a) + K\alpha^2 \right\} |n\rangle$$

$$= \left\{ \frac{\hbar\omega}{2} \left[\sqrt{n+1}\sqrt{n+1} + \sqrt{n}\sqrt{n} \right] + K\alpha^2 \right\} |n\rangle$$

$$= \left\{ \hbar\omega \left(n + \frac{1}{2} \right) + K a^2 \right\} |n\rangle$$

b. $\phi(x) = \frac{1}{\pi^{1/4}\,\Delta^{1/2}} \exp\left(-\frac{x^2}{2\Delta^2}\right)$

TISE

$$\hat{H}\,\phi(x) = E\,\phi(x)$$

$$\Rightarrow \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2 + ka^2\right]\phi(x) = \left[\hbar\omega\left(n+\tfrac{1}{2}\right) + ka^2\right]\phi(x)$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{d^2\phi(x)}{dx^2} = \left[\hbar\omega\left(n+\tfrac{1}{2}\right) - \tfrac{1}{2}m\omega^2x^2\right]\phi(x)$$

$$\Rightarrow \frac{d^2\phi}{dx^2} = \left[-\frac{2m\omega}{\hbar}\left(n+\tfrac{1}{2}\right) + \frac{m^2\omega^2}{\hbar^2}x^2\right]\phi(x)$$

$$\Rightarrow \left\{\left(-\frac{1}{2\Delta^2}\right)(+2x)\right\}^2\phi(x) = \frac{4x^2}{4\Delta^4}\phi(x) = \left[-\frac{2m\omega}{\hbar}\left(n+\tfrac{1}{2}\right) + \frac{m\omega^2}{\hbar^2}x^2\right]\phi(x)$$

$$\Rightarrow \frac{x^2}{\Delta^4}\phi(x)$$

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PROBLEM 4

A beam of spin-1/2 particles traveling in the y direction is sent through a Stern-Gerlach apparatus in which the magnetic field is inhomgeneous in the z direction, with $g\,\mu_B\,\partial B/\partial z < 0$. Here g is the g-factor of the particles, μ_B the Bohr magneton, and B the magnetic field. Two beams emerge from the apparatus. The beam that emerges with a velocity whose z component is positive (the beam traveling upward) enters a second Stern-Gerlach apparatus. The inhomgeneity in this second magnet is aligned along the unit vector $\hat{\mathbf{e}} = \sin\boldsymbol{\theta}\,\hat{\mathbf{x}} + \cos\boldsymbol{\theta}\,\hat{\mathbf{y}}$

[a] (2pts) Derive an expression for the elements of the 2×2 matrix

$$\mathbf{S} = (\hbar/2)\boldsymbol{\sigma}\cdot\hat{\mathbf{e}} = \frac{\hbar}{2}(\sigma_x e_x + \sigma_y e_y + \sigma_z e_z)$$

where σ_x, σ_y, and σ_z are the Pauli Spin matrices. Expression each matrix element in terms of $\hbar$ and the angle θ of the unit vector $\hat{\mathbf{e}}$.

[b] (2pts) What are the eigenvalues of $\mathbf{S}$? Justify your answer.

[c] (3pts) Determine the normalized eigenvectors of the matrix $\mathbf{S}$.

[d] (3pts) Derive expressions for the relative probability that the particles will be deflected into each of the two beams that emerge from the second Stern-Gerlach apparatus.

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PROBLEM 5

Consider a particle of mass m and charge e which is in perpendicular electric and magnetic fields: $\vec{E} = E\,\hat{z}$, $\vec{B} = B\,\hat{y}$. The Hamiltonian for this system is given by

$$H = \frac{1}{2m}\left((p_x - \frac{eBz}{c})^2 + p_y^2 + p_z^2\right) - eEz$$

[a] (6pts) Find the eigenvalues of this system. (Hint: Exploit the method of separation of variables, writing the eigenfunction as $\phi(\vec{r}) = \psi(z)e^{i(k_x x + k_y y)}$ where k_x and k_y are constants.)

[b] (4pts) Find the average speed in the x direction for any eigenstate.

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PROBLEM 6

Two spinless particles of mass m_1 and m_2 have a reduced mass $m = m_1 m_2/(m_1 + m_2)$. One has a charge e and the other $-e$. Together they form a hydrogen-like atom.

[a] (1 pt) Ignoring spin, the ground state wave function of the atom is given by

$$\phi_{n\ell m} = \phi_{100} = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_o}\right)^{3/2} \exp(-\frac{r}{a_o})$$

whereas the wave functions of the $n = 2$ levels are given by

$$\begin{aligned} \phi_{200} &= \frac{1}{(8\pi a_o^3)^{1/2}}\left(1 - \frac{r}{2a_o}\right)\exp[-\frac{r}{2a_o}] \\ \phi_{21m} &= \frac{1}{(24 a_o^3)^{1/2}}\left(\frac{r}{a_o}\right)\exp[-\frac{r}{2a_o}]\, Y_{1m}(\theta,\phi), \quad m = -1, 0, 1. \end{aligned}$$

Use the time-independent form of Schrodinger's equation for this hydrogen-like atom and the ground-state wave function to derive expressions for a_o and the ground-state energy E_{100}. Write your expressions terms of e, m, and fundamental constants.

22. don't know

[b] (1 pt) What is the difference in energy ΔE between the $n = 2$ and $n = 1$ states?

[c] (2pts) The 25 integrals of the form

$$\langle \phi_{n\ell m} | \vec{r} | \phi_{n'\ell' m'} \rangle$$

can be constructed from the five states listed in part [a]. Use parity arguments to determine which of these integrals are zero.

[d] (3pts) By writing

$$\begin{aligned} \vec{r} &= r\,(\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}) \\ &= r\sqrt{\frac{4\pi}{3}}\left[\left(\frac{Y_{1-1}^* - Y_{11}^*}{\sqrt{2}}\right)\hat{x} + \left(\frac{Y_{1-1}^* + Y_{11}^*}{\sqrt{2}i}\right)\hat{y} + Y_{10}^*\,\hat{z}\right] \end{aligned}$$

evaluate the integrals $\langle \varphi_{2\ell m} | \vec{r} | \phi_{2\ell' m'} \rangle$ that are not zero. (You do not have to evaluate integrals involving the $n = 1$ state.)

[e] [1 pt] Now assume a field $E_o\,\hat{z}$ is applied to the system with $eE_o a_o <<$ ΔE. In this case, the perturbation of the ϕ_{100} state is expected to be very small. Why?

[f] [2 pts] Derive expressions for the energies of the system. Consider only energies that correspond to $n = 2$ states when the field strength is zero. Discuss in words and a sketch the degeneracy (if an) of these eigenstates when the field strength is nonzero, $E_o > 0$.

P6

a. $$\frac{d^2\phi_{100}}{dr^2} + \frac{2m}{\hbar^2}\left[E_{100} + \frac{e^2}{r}\right]\phi_{100} = 0$$

$$\Rightarrow \frac{1}{\sqrt{\pi a_0^3}}\left(-\frac{1}{a_0}\right)^2 e^{-r/a_0} + \frac{2m}{\hbar^2}\left[E_{100} + \frac{e^2}{r}\right]\phi_{100} = 0$$

$$\Rightarrow \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{7/2} \frac{1}{a_0^2} + \frac{2m}{\hbar^2}\left[E_{100} + \frac{e^2}{r}\right] = 0$$

$$\Rightarrow \quad E_{100} = -\frac{\hbar}{2ma_0^2} - \frac{e^2}{r}$$